Sam's Maths Dictionary

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January 12, 2025

Introduction

Welcome to Sam's Maths Dictionary! This is a compiled pdf version of my dictionary, which is maintain on my website here: http://samgunatilleke.co.uk/dictionary.
This 'dictionary' is a list of mathematical terms and my attempts at quick, no-nonsense definitions for them. Similar to the glossary, this is primarily a tool for my own use and many of the definitions will reflect this. Unlike the glossary, this resource is not supposed to give any explanations or examples beyond the definitions themselves - a group on this page is just a set with a binary operations which is associative, unital, and invertible. The web version of this resource has separate pages for each entry and is rendered using MathJax, while this version is a pdf file produced using LaTeX- there should not be any compatibility issues in the typesetting, but do let me know if you notice any.

!#123

2-Category

A 2-Category $\mathcal{C}$ is a category which is enriched over **Cat**, i.e for each pair of objects $x, y \in \mathcal{C}_0$, the hom-set $\mathcal{C}(x, y)$ forms a category, i.e. there is a collection $\mathcal{C}_2$ of 2-arrows, each of which has a domain and codomain a pair of parallel 1-arrows in $\mathcal{C}_1$, and these behave nicely under composition of 1-arrows. In particular, for 1-arrows $f, g : X \to Y$ and $h, k : Y \to Z$, with 2-arrows $\eta : f \Rightarrow g$ and $\xi : h \Rightarrow k$, there is a composite 2-arrow $(\xi * \eta) : (h \circ f) \Rightarrow (k \circ g)$.

A

Abelian Group**

A group $(G, \cdot)$ is 'abelian' if the group operation is commutative, i.e. $\forall g, h \in G : g \cdot h = h \cdot g$.

Abelian Subalgebra**

Given a Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is called abelian if it has a trivial bracket with itself: $\forall x, y \in \mathfrak{h} : [x, y] = 0$. Equivalently, the subalgebra product of $\mathfrak{h}$ with itself is trivial, $[\mathfrak{h}, \mathfrak{h}] = 0$. If all of $\mathfrak{g}$ is an abelian subalgebra, we call $\mathfrak{g}$ abelian.

Algebra**

An an algebra $(\mathcal{A}, \diamond)$ over a ring R is an R-module together with an R-bilinear binary product $\diamond : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ - i.e. $forall x, y, z, w \in \mathcal{A} : \forall r, s \in R : (r \cdot x + y) \diamond (s \cdot z + w) = (rs) \cdot (x \diamond z) + r \cdot (x \diamond w) + s \cdot (y + z) + y \diamond w$. Typically one specifies the case where R is a field and $\diamond$ is associative.

Alternating**

Given an abelian group $(A, +)$ and a binary product $[\cdot, \cdot] : A \times A \to A$ (typically $(A, +)$ has the additional structure of being a vector space and $[\cdot, \cdot]$ is also bilinear, but this is not required), one says that $[\cdot, \cdot]$ is alternating if $\forall x \in A : [x, x] = 0$.

Associative**

Given a binary product $\cdot : X \times X \to X$ on a set X, one says that $\cdot$ is associative if the order of calculation does not matter for repeated applications of $\cdot$, i.e. $\forall x, y, z \in X : x \cdot (y \cdot z) = (x \cdot y) \cdot z$. More generally if $\circ$ is a partial function on X, $\circ$ is called associative if $f \circ (g \circ h) = (f \circ g) \circ h$ whenever $f, g, h \in X$ such that $f \circ g$, $g \circ h$, $f \circ (g \circ h)$, $(f \circ g) \circ h$ exist, for example composition of arrows in a category. Finally, given multiple sets and functions between them, $\cdot : A \times A \to A$ and $\star : A \times B \to B$, one says that $\cdot, \star$ associate with each other if $\forall \alpha, \beta \in A : \forall x \in B : \alpha \star (\beta \star x) = (\alpha \cdot \beta) \star x$, for example multiplication in the field and scalar multiplication for a vector space.

Axiom of Choice**

An axiom of set theory, typically included alongside the Zermelo-Fraenkel axioms (from which it is independent) as one of the foundational axioms of set theory. Its statement is ''For any collection of nonempty, disjoint, there exists a set which contains precisely each element from each member of the collection." In symbols: $\forall x : (\emptyset \notin x \wedge \forall a, b \in x : a \cap b = \emptyset) \Rightarrow \exists c : \forall a \in x : \exists! y \in a : y \in c$. The axiom of choice is equivalent to Zorn's Lemma, and statements such as ''Every set has a total order", ''every vector space has a basis".

Axiom of Extensionality**

One of the Zermelo-Fraenkel axioms of set theory, which states that two sets are equal precisely if they share all their elements: $\forall x : \forall y : x = y \Leftrightarrow (\forall a : a \in x \Leftrightarrow a \in y)$. This axiom means that, for example, $\{x, y\} = \{y, x\} = \{x, x, y\}$ i.e. ordering and repeats do not affect the definition of a set.

Axiom of Foundation**

One of the Zermelo-Fraenkel axioms of set theory, which states that every non-empty set x contains an element which is disjoint from x: $\forall x : x \neq \emptyset \Rightarrow (\exists a \in x : a \cap x = \emptyset)$. Important consequences are that no set may contain itself, nor can there exist 'cycles' in the epsilon relation (i.e. a sequence of sets $x_1, ..., x_n$ satisfying $x_1 \in x_2 \in \cdots \in x_n \in x_1$, nor can there exist a chain of 'infinite descent' (an infinite sequence $x_1, x_2, ...$ such that $x_1 \ni x_2 \ni \cdots$).

Axiom of Infinity**

One of the Zermelo-Fraenkel axioms of set theory, which asserts the existence of an infinite set, which has the same cardinality as the natural numbers. $\exists I : (\emptyset \in I \wedge \forall x : x \in I \Rightarrow x \cup \{x\} \in I$. As such, I contains $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, ...$ and so on.

Axiom of Replacement**

One of the Zermelo-Fraenkel axioms of set theory, which states that whenever x is a set and $\mathscr{R}$ is a functional relation on x, the image of x under $\mathscr{R}$ (the set of all b such that $a\mathscr{R}b$ for some $a \in x$) is a set: $\forall x : [\forall a \in x : \exists! b : a\mathscr{R}b] \Rightarrow [\exists y : \forall b : b \in y \Leftrightarrow \exists a \in x : a\mathscr{R}b]$. This is actually an axiom schema (as it holds for every relation $\mathscr{R}$) and underlies the principle of restricted comprehension.

Axiom of the Pair Set**

An axiom of Zermelo-Fraenkel set theory which states that, for any two sets X, Y, there exists a set $(\{X, Y\})$ whose elements are precisely X and Y: $\forall X : \forall Y : \exists Z : \forall a : (a \in Z \Leftrightarrow a = X \vee a = Y)$.

B

Banach Space

A Banach space $(\mathcal{B}, ||\cdot||)$ is a normed vector space which is Cauchy complete with respect to the metric induced by the norm: $d(x, y) = ||x - y||$.

Basis (Linear Algebra)**

A basis of a vector space $(V, +, \cdot)$ over a field k is a subset $S \subseteq V$ which is linearly independent and spans V. In infinite dimensions, one specifies a difference between a 'Hamel basis' (using finite sums for linear independence and span) and a 'Schauder basis' (using convergent infinite sums for linear independence and span).

Binary Product**

Given a set X, a binary product (or 'binary operation') on X is a function $\cdot : X \times X \to X$, typically written $\cdot : (x, y) \mapsto \cdot(x, y) = x \cdot y)$.

Bound (Order Theory)**

Given a partial order $(P, \leq)$ and a subset $S \subseteq P$, an upper (resp. lower) bound of S is an element $x \in P$ such that $\forall s \in S : s \leq x$ (resp. $x \leq s$).

Bounded Set**

Given a metric space (X, d), a subset $A \subseteq X$ is called 'bounded' if it has a finite diameter, i.e. $\exists r \in \mathbb{R} : \forall x, y \in A : d(x, y) \leq r$.

Bundle (Topology)**

A 'bundle' (E, p, B) consists of sets (or, usually, topological spaces) E (the 'total space') and B (the 'base space'), and a map (or a continuous map, for topological spaces) $p : E \to B$.

C

Cartesian Product**

Given sets X, Y, their Cartesian product $X \times Y$ consists of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$ (by 'ordered pair' one means that $(x, y) = (x', y') \Leftrightarrow (x = x' \wedge y = y')$). This set is constructible within theZermelo-Fraenkel axioms of set theory by identifying the set $\{\{x\}, \{x, y\}\}$ with the pair (x, y), so that $X \times Y = \{S \in \mathcal{P}(\mathcal{P}(X \cup Y)) | \exists x \in X : \exists y \in Y : S = \{\{x\}, \{x, y\}\}\}$.

Category**

A category $C = (C_0, C_1, \mathrm{dom}, \mathrm{cod}, 1, \circ)$ consists of a collection (formally a member of some Grothendieck universe) C_0 of 'objects', a collection C_1 of 'arrows' (or 'morphisms'), functions $\mathrm{dom}, \mathrm{cod} : C_1 \to C_0$ which assign to each arrow its domain (source) and codomain (target) respectively, a function $1 : C_0 \to C_1, X \mapsto 1_X$ which assigns to each object an 'identity arrow', and a partial function $\circ : C_1 \times C_1 \to C_1, (f, g) \mapsto f \circ g$ so that $f \circ g$ is defined precisely when $\mathrm{dom}(f) = \mathrm{cod}(g)$, 'function composition'. The identity map must satisfy $\forall X \in C_0 : \mathrm{dom}(1_X) = \mathrm{cod}(1_X) = X$, and the composition (where it is defined) must satisfy $\mathrm{dom}(f \circ g) = \mathrm{dom}(g)$ and $\mathrm{cod}(f \circ g) = \mathrm{cod}(f)$. Composition must be associative, so that $\forall f, g, h \in C_1 : (\mathrm{dom}(f) = \mathrm{cod}(g) \wedge \mathrm{dom}(g) = \mathrm{cod}(h)) \Rightarrow f \circ (g \circ h) = (f \circ g) \circ h$, and unital via the identity, i.e. $\forall X \in C_0 : \forall f \in C_1 : (\mathrm{dom}(f) = X \Rightarrow f \circ 1_X = f) \wedge (\mathrm{cod}(f) = X \Rightarrow 1_X \circ f = f)$.

Cauchy Complete**

A metric space metric space (X, d) is 'Cauchy complete' if every Cauchy sequence in X has a limit in X. For example: $\mathbb{R}$ is Cauchy complete by definition, while $\mathbb{Q}$ is not, as the sequence $\sum_{r=0}^{n} \frac{1}{n!}$ is Cauchy but its limit, e, is not in $\mathbb{Q}$. Given a metric space (X, d), its Cauchy (or metric) completion $(\bar{X}, \bar{d})$ is the quotient of the set of all Cauchy sequences in X modulo the equivalence relation $(a_i)_{i=0}^{\infty} \sim (b_i)_{i=0}^{\infty} :\Leftrightarrow \lim_{i \to \infty} d(a_i, b_i) = 0$, so that two sequences are equivalent if they share a limit point, with an induced metric $\bar{d}\left([(a_i)_{i=0}^{\infty}], [(b_i)_{i=0}^{\infty}]\right) := \lim_{i \to \infty} d(a_i, b_i)$. $\mathbb{R}$ is the metric completion of $\mathbb{Q}$.

Cauchy Sequence**

Given a metric space (X, d), a sequence $(a_i)_{i=0}^{\infty}$ is 'Cauchy' if the diameter of the partial sequence $(a_i)_{i=n}^{\infty}$ tends to 0 as $n \to \infty$, i.e. $\forall \epsilon > 0 : \exists N \geq 0 : \forall n, m \geq N : d(a_n, a_m) < \epsilon$ - for any given radius, there exists a point in the sequence beyond which all points are within that radius of each other.

Chain (Order Theory)**

A chain of a partial order $(P, \leq)$ is a subset $S \subseteq P$ such that $(S, \leq)$ is a total order.

Chain Complex**

Given an abelian category $\mathcal{A}$, a chain complex with coefficients in $\mathcal{A}$ $(C_\bullet)$ is a sequence $(C_n)_{n\in\mathbb{Z}}$ of objects of $\mathcal{A}$ (the 'n-chains' of $C_\bullet$) with a collection of maps $d_n : C_n \to C_{n-1}$ (the 'differential' of $C_\bullet$) such that $d_{n-1} \circ d_n = 0$ for each $n \in \mathbb{Z}$. The kernels and images of the differential, $Z_n = \ker(d_n) \subseteq C_n$ and $B_n = \ker(d_{n+1}) \subseteq C_n$ are called the n-cycles and n-boundaries respectively, and the quotient $H_n = Z_n/B_n$ is called the n^{th} homology object of $C_\bullet$.

Class**

Within Zermelo-Fraenkel set theory (and other axiomatisations), a class is a collection of sets satisfying some predicate, for example $\{x|R(x)\}$, where R is a predicate which identifies x as describing a ring. Classes are often not sets as they are 'too large', in which case the class is called 'proper' - the aforementioned collection of all rings is an example of a proper class.

Closed Ball**

Given a metric space (X, d), a point $x \in X$ and a radius $r \in \mathbb{R}_{>0}$, the closed ball about x of radius r is the subset of all points of X at a distance at most r from x; $\bar{B}_r(x) := \{y \in X | d(x, y) \leq r\}$.

Closed Set**

Given a topological space $(X, \mathcal{T})$, a subset $A \subseteq X$ is 'closed' (with respect to the topology $\mathcal{T}$) if its complement is open, i.e. $X \backslash A \in \mathcal{T}$. The definition of a topology means that $\emptyset$ and X are always closed, as is the union of a finite number of closed sets and the intersection of an arbitrary number of closed sets.

Coarse/Fine (Topology)**

Given a set X and two topologies $\mathcal{T}, \mathcal{T}'$ on X, one says that $\mathcal{T}$ is coarser than $\mathcal{T}'$ (or, equivalently, $\mathcal{T}'$ is finer than $\mathcal{T}$) if $\mathcal{T} \subseteq \mathcal{T}'$. This turns the set of all topologies on X into a partial order. One often seeks the coarsest topology which satisfies a given condition, for example the product topology is the coarsest topology on $X \times Y$ which makes both $\pi_1 : X \times Y \to X$, $(x, y) \mapsto x$ and $\pi_2 : X \times Y \to Y$, $(x, y) \mapsto y$ continuous. The coarsest topology on any set X is $\{\emptyset, X\}$ (the 'trivial' or 'indiscrete' topology), while the finest is $\mathcal{P}(X)$ (the 'discrete' topology).

Co-Cone Category**

Given a (typically small) category $\mathcal{I}$, a category $\mathcal{C}$, and a functor $F : \mathcal{I} \to \mathcal{C}$, a co-cone under F is an object $X \in \mathcal{C}_0$ with a collection of arrows to X from the image of each object of $\mathcal{I}$ under F, $\{(f_i : F(i) \to X) \in \mathcal{C}_1 | i \in \mathcal{I}_0\}$, which is compatible with arrows in $\mathcal{I}$ i.e. $\forall (g : i \to j) \in \mathcal{I}_1 : f_i = F_j \circ F(g)$. Given two co-cones $(X, \{f_i\})$ and $(Y, \{g_i\})$, an arrow between them is some $(u : X \to Y) \in \mathcal{C}_0$ which respects the mappings from F, i.e. $\forall i \in \mathcal{I}_0 : f_i = u \circ g_i$. The collection of all co-cones over F with arrows between them thus forms a category, $(F \downarrow \Delta$, an example of a comma category.

Cokernel**

Given an arrow $f : X \to Y$ in an additive category $\mathcal{A}$, its cokernel is its coequaliser with the zero map $0 : X \to Y$, i.e. an object $\mathrm{coker}(f) \in \mathcal{A}_0$ together with an arrow $\mathrm{coker}(f) : Y \to \mathrm{coker}(f)$ satisfying the universal property that, for $g : Y \to Z$ any arrow in $\mathcal{A}$ such that $g \circ f = 0$, there is a unique arrow $\bar{g} : \mathrm{coker}(f) \to Z$ such that $g = \bar{g} \circ \mathrm{coker}(f)$, i.e. any map which vanishes upon precomposition with f factors through the cokernel. In the category of abelian groups, for example, the cokernel of a map is the quotient of the codomain by the image (the arrow is simply the canonical projection $Y \to Y/\mathrm{im}f$).

Commutative**

A binary operation $*$ on a set X is called commutative if the order of arguments is irrelevant, i.e. $\forall x, y \in X : x * y = y * x$.

Commutative Ring**

A ring $(R, +, \times)$ is 'commutative' if its multiplication is commutative, i.e. $\forall a, b \in R : a \times b = b \times a$.

Commutator Lie Algebra**

Given an associative algebra $\mathfrak{A}, \diamond$ over a ring R, the commutator Lie algebra of $\mathfrak{A}$ is the Lie algebra $\mathfrak{A}, [\cdot,\cdot]_\diamond$ with bracket $[x, y]_\diamond := x \diamond y - y \diamond x$. The mapping of an associative algebra to its commutator algebra is functorial and is right adjoint to the functor associating an R-Lie algebra to its universal enveloping algebra.

Compact (Topology)**

A topological space X is 'compact' if every open cover $(U)_{i\in I}$ of X admits a finite subcover $(U_{i_n})_{n=1}^N$. A subset A of X is a compact subet if A is compact under the subspace topology. In a Heine-Borel metric space (such as, but not limited to, $\mathbb{R}^n$), a subset is compact iff it is closed and bounded.

Complement**

Given a subset $A \subseteq X$ of a set X, the complement of A in X is the set of all elements of X which are not elements of A, i.e. $X \backslash A := \{x \in X | x \notin A\}$.

Cone Category**

Given a (typically small) category $\mathcal{I}$, a category $\mathcal{C}$, and a functor $F : \mathcal{I} \to \mathcal{C}$, a cone over F is an object $X \in \mathcal{C}_0$ with a collection of arrows from X to the image of each object of $\mathcal{I}$ under F, $\{(f_i : X \to F(i)) \in \mathcal{C}_1 | i \in \mathcal{I}_0\}$, which is compatible with arrows in $\mathcal{I}$ i.e. $\forall (g : i \to j) \in \mathcal{I}_1 : f_j = F(g) \circ f_i$. Given two cones $(X, \{f_i\})$ and $(Y, \{g_i\})$, an arrow between them is some $(u : X \to Y) \in \mathcal{C}_0$ which respects the mappings into F, i.e. $\forall i \in \mathcal{I}_0 : f_i = g_i \circ u$. The collection of all cones over F with arrows between them thus forms a category, $(\Delta \downarrow F$, an example of a comma category.

Continuous (Topology)**

Given topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a function $f : X \to Y$ is called 'continuous' if whenever $U \in Y$ is open, so is its preimage $f^{-1}(U) \subseteq X$, i.e. $\forall U \in \mathcal{T}_Y : f^{-1}(U) \in \mathcal{T}_X$.

D

Diameter (Metric Space)**

Given a metric space (X, d), the diameter of a subset $A \subseteq X$ is the supremum of distances between pairs of points in A, i.e. $\mathrm{diam}(A) = \sup_{x,y\in A} d(x, y)$.

Differentiable Atlas

A differentiable atlas of degree k of a manifold $(X, \mathcal{T})$ is a collection of ordered pairs ('charts') $\mathcal{A} = (U_i, \phi_i)_{i \in I}$ where $U_i \in \mathcal{T}$ and $\phi_i : U_i \to \mathbb{R}^n$ is a homeomorphism onto its image $\phi_i(U_i) \in \mathcal{T}_{\mathbb{R}^n}$ (where $\mathcal{T}_{\mathbb{R}^n}$ is the metric topology on $\mathbb{R}^n$). The charts must also be C^k compatible, i.e. for every pair of charts (U, ϕ) and (V, ψ), the transition map $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ must be C^k differentiable. The set of all degree-k atlases on X form a partial order, and a maximal element of this partial order (i.e. an atlas $\mathcal{A}$ such that no other chart on X is compatible with every chart of $\mathcal{A}$) is called a C^k-differentiable structure on X.

Disjoint**

Two sets A, B are disjoint if their intersection is empty, $A \cap B = \emptyset$.

E

(Axiom of the) Empty Set**

The empty set $\emptyset$ is a set which contains no elements; $\forall x : x \notin \emptyset$. The axiom of the empty set, one of the axioms of Zermelo-Fraenkel set theory, asserts the existence of such a set. By the axiom of extensionality the empty set is unique, and by definition is a subset of every set.

Equaliser**

Given a pair of parallel arrows $f, g : X \to Y$ in a category $\mathcal{C}$, their equaliser is an object $\mathrm{Eq}(f, g) \in \mathcal{C}_0$ with an arrow $u : \mathrm{Eq}(f, g) \to X$ such that $f \circ u = g \circ u$ and satisfying the universal property that, whenever there is an arrow $h : Z \to X$ such that $f \circ h = g \circ h$, there is a unique arrow $\bar{h} : Z \to \mathrm{Eq}(f, g)$ such that $h = u \circ \bar{h}$ - any map which makes f, g equal upon precomposition factors through the equaliser. This is an example of a limit in category theory. In **set**, the equaliser of two maps $f, g : X \to Y$ is the subset $\mathrm{Eq}(f, g) = \{x \in X | f(x) = g(x)\}$ and the arrow u is just its inclusion into X.

Equivalence Relation**

A relation $\sim$ is called an equivalence relation on a set X if it is reflexive, symmetric and transitive, i.e. $\forall x \in X : x \sim x$, $\forall x, y \in X : x \sim y \Leftrightarrow y \sim x$, and $\forall x, y, z \in X : (x \sim y \wedge y \sim z) \Rightarrow x \sim z$. Given an element $x \in X$, its 'equivalence class' is the subset $[x] := \{y \in X | x \sim y\}$, sometimes denoted $[x]_\sim$. The equivalence classes of elements of X under $\sim$ form a partition of X.

F

Fiber Bundle**

A fiber bundle (E, B, p, F) is a bundle of topological spaces (E, B, p) along with a third topological space F (the 'fiber') such that $\forall x \in B : p^{-1}(B) \cong F$, i.e. the preimage of each point of the base-space under projection is homeomorphic to the fiber, and also for each $x \in B$, $\exists U \in \mathbf{Top}(B) : \exists(\phi : p^{-1}(U) \to U \times F) \in \mathbf{Top} : p = \pi_U \circ \phi$ - there is an open neighbourhood U of every point x of the base space, upon the preimage of which the projection factors through the projection onto U from the trivial product $U \times F$.

Field**

A field is a ring with commutative and invertible multiplication; The field $(k, +, \times)$ consists of a set k and binary operations $+, \times : k \times k \to k$ which are both commutative, associative, and unital (with units $0_k, 1_k$ respectively), such that $\forall x \in k : \exists(-x) \in k : x + (-x) = 0_k$, $\forall x \in k \backslash \{0_k\} : \exists x^{-1} \in k : x \times x^{-1} = 1_k$, and $\times$ distributes over $+$, i.e. $\forall a, b, c \in k : a \times (b + c) = (a \times b) + (a \times c)$. Typically it is also required that $0_k \neq 1_k$.

Fréchet Space (Functional Analysis)**

A Fréchet space is a topological vector space $(V, \mathcal{T}, +, \cdot)$ such that the topological space $(V, \mathcal{T})$ is metrizable (and so also Hausdorff), V is Cauchy complete with respect to this metric, and V is locally convex.

Function**

Given sets X, Y, a function $f : X \to Y$ is an assignment of precisely one element $f(x)$ of Y to each element x of X, written $x \mapsto f(x)$. This is equivalent to there being a functional relation $\mathscr{R}$ on X such that the image lies within Y; $\forall x \in X : \exists! y \in Y : x\mathscr{R}y$, with $y = f(x) \Leftrightarrow x\mathscr{R}y$.

Functional Relation**

A relation $\mathscr{R}$ is called 'functional' on a set x if for each element $a \in x$ there is a unique b such that $a\mathscr{R}b$, i.e. $\forall a \in x : \exists! b : a\mathscr{R}b$.

Functor**

Given categories $\mathcal{C}$, $\mathcal{D}$, a (covariant) functor $F : \mathcal{C} \to \mathcal{D}$ is a pair of maps (typically both simply labelled F) $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ and $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ which are compatible with identities and compositions, i.e. $\forall X \in \mathcal{C}_0 : F_1(1_X) = 1_{F_0(X)}$ and $\forall X \xrightarrow{f} Y \xrightarrow{g} Z \in \mathcal{C} : F_1(g \circ f) = F_1(g) \circ F_1(f)$.

G

Graph (of a function or relation)**

Given a function $f : X \to Y$ between sets X, Y, its graph $\Gamma_f \subseteq X \times Y$ is the subset of their Cartesian product given by all pairs $(x, f(x))$, i.e. $\Gamma_f := \{(x, y) \in X \times Y | x \in x \land y = f(x)\}$. More generally, given a relation $\mathscr{R}$, its graph is $\Gamma_{\mathscr{R}} := \{(x, y) | x\mathscr{R}y\}$. If $\mathscr{R}$ is functional or is a relation between sets, then $\Gamma_{\mathscr{R}}$ is a set, but in general it may be a proper class.

Group**

A group $(G, \cdot)$ is a set G equipped with a binary product $\cdot : G \times G \to G$ which is associative, has a unit $e \in G$, and has inverses.

H

Hausdorff Space**

A topological space $(X, \mathcal{T})$ is called Hausdorff if any two distinct points are contained in disjoint neighbourhoods; $\forall x, y \in X : x \neq y \Rightarrow (\exists U, V \in \mathcal{T} : x \in U \land y \in V \land U \cap V = \emptyset)$.

Hom Set**

Given a category $\mathcal{C}$ and a pair of objects $x, y \in \mathcal{C}_0$, the 'hom set' $\mathcal{C}(x, y) = \mathrm{Hom}_{\mathcal{C}}(x, y)$ is the collection of all arrows from x to y, i.e. $\{f \in \mathcal{C}_1 | \mathrm{dom}(f) = x \land \mathrm{cod}(f) = y\}$.

Homeomorphism

Given topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a function $f : X \to Y$ is called a 'homeomorphic' if f is bijective and continuous, and its inverse $f^{-1} : Y \to X$ is also continuous. If such a map exists then one writes $X \cong_{\mathbf{Top}} Y$ and calls X and Y 'homeomorphic', the topological word for isomorphic. One says that $f : X \to Y$ is 'homeomorphic onto its image' if $\tilde{f} : X \to \mathrm{im}(f)$ is a homeomorphism when $\mathrm{im}(f)$ is endowed with the subspace topology.

Horizontal Composition**

Given three categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$, pairs of parallel functors $F, G : \mathcal{A} \to \mathcal{B}$ and $H, K : \mathcal{B} \to \mathcal{C}$ and two natural transformations $\eta : F \Rightarrow G$, $\xi : H \Rightarrow K$, their horizontal composition is the natural transformation $(\xi * \eta) : H \circ F \Rightarrow K \circ G$ given by $(\xi * \eta)_x = \xi_{G(x)} \circ H(\eta_x) : HF(x) \to KG(x)$.

I

Ideal (Lie Algebra)**

Given a Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is called an ideal if $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$, where $[\mathfrak{h}, \mathfrak{g}]$ is the standard product of subalgebras. The subalgebras $\mathfrak{g}, \{0\}$ are ideals for any Lie algebra $\mathfrak{g}$ and so are called 'trivial'.

Ideal (Ring Theory)**

Given a ring $(R, +, \times)$, a subset $I \subseteq R$ such that $(I, +)$ is a subgroup of $(R, +)$ and $RI = \{ra | r \in R, a \in I\} \subseteq I$ (resp. $IR = \{ar | r \in R, a \in I\} \subseteq I$) is called a left (resp. right) ideal of R. If I is both a left and right ideal then we call it a two-sided ideal and write $I \triangleleft R$. An ideal is called 'proper' if it is not R itself, and 'nontrivial' if it is not the zero ideal $0 = \{0_R\}$.

Image (Homological Algebra)**

Given an arrow $f : X \to Y$ in an additive category $\mathcal{A}$, its image is the kernel of its cokernel, while its coimage is the cokernel of its kernel - in an abelian category these two objects are isomorphic.

Infimum**

Given a Partial Order $(P, \leq)$ and a subset $A \subseteq P$, the infimum of A, if it exists, is the greatest lower bound in P of A, i.e. the element $\inf A \in P$ such that $\forall x \in P : (\forall a \in A : x \leq a \Rightarrow x \leq \inf A)$. If the infimum exists it must be unique.

Initial Object**

An object ι of a category $\mathcal{C}$ is called 'initial' if, for any other object x of $\mathcal{C}$, there is a unique arrow from ι to x, i.e. $\forall x \in \mathcal{C}_0 : \exists! \iota_x \in \mathcal{C}_1 : \text{dom}(\iota_x) = \iota \wedge \text{cod}(\iota_x) = x$ or (in terms of hom-sets) $\forall x \in \mathcal{C}_0 : \text{Hom}_{\mathcal{C}}(\iota, x) = \{\iota_x\}$. If a category has initial objects they are all isomorphic to each other.

Intersection**

Given a set X, its intersection $\cap X$ is the set which contains all elements contained in all sets in X: $\forall a : a \in \cap X \iff \forall x \in X : a \in x$, or equivalently $\cap X = \{a | \forall x \in X : a \in x\}$. When $X = \{x_1, ..., x_n\}$ is finite, we typically write $\cap X = x_1 \cap \cdots \cap x_n$, and when $X = \{x_i | i \in I\}$ is an indexed collection of sets we write $\cap X = \cap_{i \in I} x_i$. The existence of the intersection of X is guaranteed by the axiom of replacement.

Isomorphism**

Given a category $\mathcal{C}$, an arrow $f : X \to Y$ in $\mathcal{C}$ is called an isomorphism (or 'invertible') if there exists an arrow $\tilde{f} : Y \to X$ such that $f \circ \tilde{f} = 1_Y$ and $\tilde{f} \circ f = 1_X$. If such an $\tilde{f}$ exists then it is necessarily unique and generally written f^{-1}, and X and Y are called 'isomorphic'. Isomorphism is an equivalence relation on the objects of $\mathcal{C}$, and means that X and Y are essentially the same (from the point of view of the category).

J

Jacobi Identity**

Given an abelian group $(A, +)$ and a binary product $[\cdot, \cdot] : A \times A \to A$ (typically $(A, +)$ has the additional structure of being a vector space and $[\cdot, \cdot]$ is also bilinear, but this is not required), one says that $[\cdot, \cdot]$ obeys the Jacobi identity if $\forall x, y, z \in A : \big[x, [y, z]\big] + \big[y, [z, x]\big] + \big[z, [x, y]\big] = 0$.

K

Kernel**

In an additive category $\mathcal{A}$, the 'kernel' of an arrow $f : X \to Y$ refers to both an object $\ker(f) \in \mathcal{A}_0$ and an arrow $(\ker(f) : \ker(f) \to X) \in \mathcal{A}_1$. The kernel of f is the equaliser between f and the 0-map from X to Y, the equaliser being a specific example of a limit: given any object $Z \in \mathcal{A}_0$ and any arrow $(g : Z \to X) \in \mathcal{A}_1$ such that $f \circ g = 0 \circ g = 0$, there exists a unique arrow $\tilde{g} : Z \to \ker(f)$ such that $g = \ker(f) \circ \tilde{g}$ - any map which vanishes upon composition with f factors through the kernel of f.

L

Lie Algebra**

A Lie algebra over a ring R is a left R-module $\mathfrak{g}$ equipped with an R-bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is alternating ($\forall x \in \mathfrak{g} : [x, x] = 0$) and satisfies the Jacobi Identity ($\forall x, y, z \in \mathfrak{g} : \big[x, [y, z]\big] + \big[z, [x, y]\big] + \big[y, [z, x]\big] = 0$).

Limit (Category Theory)**

Given a (typically small) category $\mathcal{I}$, a category $\mathcal{C}$, and a functor $F : \mathcal{I} \to \mathcal{C}$, a limit of F is an initial object in the category of cones over F, i.e. an object $\lim_{\mathcal{I}} F \in \mathcal{C}_0$ and a collection of arrows $\{(\phi_i : \lim_{\mathcal{I}} F \to F(i)) \in \mathcal{C}_1 | i \in \mathcal{I}_0\}$, such that $\forall (f : i \to j) \in \mathcal{I}_1 : \phi_j = F(f) \circ \phi_i$, with the universal property that, whenever there is an object $X \in \mathcal{C}_0$ with arrows $\{(g_i : \lim_{\mathcal{I}} F \to F(i)) \in \mathcal{C}_1 | i \in \mathcal{I}_0\}$ such that $\forall (f : i \to j) \in \mathcal{I}_1 : g_j = F(f) \circ g_i$, there is a unique arrow $u : \lim_{\mathcal{I}} F \to X$ such that $\forall i \in \mathcal{I}_0 : g_i = \phi_i \circ u$. Limits are a generalisation of products and equalisers and are dual to the concept of a colimit.

Limit (in a Metric Space)**

Given a metric space (X, d) and a sequence $(a_i)_{i=1}^n$ in X, a point $L \in X$ is a limit of the sequence if, for any given radius, there is a point in the sequence beyond which all terms lie within that distance of L: $\forall \epsilon > 0 : \exists N > 0 : \forall n > N : d(L, a_n) < \epsilon$. If a limit exists then it is unique, and it is also the limit in the topological sense when X is endowed with the metric topology.

Limit (in a Topological Space)**

Given a topological space $(X, \mathcal{T})$ and a sequence $(a_i)_{i=1}^n$ in X, a point $L \in X$ is a limit of the sequence if, for any open neighbourhood of L, there is a point in the sequence beyond which all terms lie within that neighbourhood: $\forall U \in \mathcal{T} : L \in U \Rightarrow (\exists N > 0 : \forall n > N : a_n \in U)$. In a general topological space, the limit is not necessarily unique.

Linearly Independent**

Given a vector space V over a field k, a subset $S \subseteq V$ is called 'linearly independent' if the only finite linear combination of elements of S to equal 0 is the trivial one, i.e. $\forall T \subseteq S : |T| < \infty \Rightarrow (\forall\, [f : T \to k,\ v \mapsto a_v] : \sum_{v \in T} a_v v = 0 \Rightarrow \forall v \in T : a_v = 0)$ - for any finite subset T of S and assignment of coefficients in k to each element of T, if the sum of these vectors with these coefficients is the zero vector then all coefficients must have been 0. In the definition of a Schauder basis, the finiteness condition on T is dropped.

Local Ring

A ring R is called 'local' if it has a unique maximal ideal (this may be maximal among left- right- or two-sided ideals, all are equivalent).

Localization (of a Ring)

The localization of a commutative ring $(R, +, \cdot)$ by a multiplicatively closed subset S containing 1 (i.e. $\forall s, t \in S : s \cdot t \in S,\ 1 \in S$) is the ring $S^{-1}R$ of all formal fractions $\left\{\frac{r}{s} \middle| r \in R,\ s \in S\right\}$, subject to the standard rules of addition and multiplication of fractions, i.e. $\frac{r_1}{s_1} + \frac{r_2}{s_2} := \frac{r_1 \cdot s_2 + r_2 \cdot s_1}{s_1 \cdot s_2}$, $\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} := \frac{r_1 \cdot r_2}{s_1 \cdot s_2}$ - in this way, the rational numbers $\mathbb{Q}$ are simply the localization of the integers $\mathbb{Z}$ by the set of non-zero integers. If $\mathfrak{p}$ is a prime ideal of R, its complement is (by definition) a multiplicatively closed subset, so one often refers to 'R localised at $\mathfrak{p}$', meaning the localization $(R \backslash \mathfrak{p})^{-1}R$.

M

Manifold

A topological space $(X, \mathcal{T})$ is called a manifold of dimension n if there exist open sets $\{U_i | i \in I\} \subseteq \mathcal{T}$ and maps $\{\phi_i : U_i \to \mathbb{R}^n | i \in I\}$ such that the (U_i) form a open cover of X, each $\phi_i(U_i)$ is open in the metric topology of $\mathbb{R}^n$, and each ϕ_i is a homeomorphism onto its image. Often one also requires that X be Hausdorff and second countable or paracompact.

Maximal Ideal

Given a ring R, an ideal $\mathfrak{m}$ of R is called 'maximal' if it is a proper ideal ($\mathfrak{m} \neq R$) which is not contained in any other proper ideal. In the case of a non-commutative ring, one considers maximal left- and maximal right- ideals.

Maximal/Minimal Element**

Given a partial order $(P, \leq)$, a maximal (resp. minimum) element of P is some $M \in P : \forall x \in P : x \leq M$ (resp. $m \in P : \forall x \in P : m \leq M$).

Metric Space**

A metric space (X, d) consists of a set X (the 'space') with a function $d : X \times X \to \mathbb{R}_{\geq 0}$ (the 'metric') such that: d is symmetric, $\forall x, y \in X : d(x, y) = d(y, x)$, non-degenerate $\forall x, y \in X : d(x, y) = 0 \Leftrightarrow x = y$, and obeys the triangle inequality: $\forall x, y, z \in X : d(x, y) + d(y, z) \geq d(x, z)$.

Metric Topology**

Given a metric space (X, d), the metric topology $\mathcal{T}_d$ on X has as its open sets precisely those sets U such that every point of U is contained in an open ball contained in U, i.e. $\mathcal{T}_d := \{U \subseteq X | \forall x \in U : \exists r_x > 0 : B_{r_x}(x) \subseteq U\}$.

Metrizable**

A topological space $(X, \mathcal{T})$ is called metrizable if the topology arises from a metric, i.e. if there exists a function $d : X \times X \to \mathbb{R}_{\geq 0}$ such that (X, d) is a metric space, and $\mathcal{T}$ is equal to the induced topology on X due to d.

Monoid**

A monoid $(M, \cdot)$ consists of a set M with a binary operation $\cdot : M \times M \to M$ which is associative and unital but, unlike for a group, not necessarily invertible. Examples include the natural numbers under addition and the integers under multiplication.

N

Natural Transformation**

Given parallel functors $F, G : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$, $\mathcal{D}$, a 'natural transformation' $\eta : F \Rightarrow G$ is an assignment of an arrow $(\eta_x : F(x) \to G(x)) \in \mathcal{D}_1$ to each object $x \in \mathcal{C}_0$, in a 'natural way', i.e.

for every arrow $(f : x \to y) \in \mathcal{C}_1$ there is a commutative square in $\mathcal{D}$: $\eta_y \circ F(f) = G(f) \circ \eta_x$. Natural transformations (combined with vertical composition turn the hom-set $\mathbf{Cat}(\mathcal{C}, \mathcal{D})$ of functors from $\mathcal{C}$ to $\mathcal{D}$ into a category of its own, while vertical and horizontal composition together turn **Cat** into a 2-category.

Neighbourhood**

Given a toplogical space $(X, \mathcal{T})$, a neighbourhood of a point $x \in X$ is a subset $A \subseteq X$ which contains an open set containing x; $existsU \in \mathcal{T} : x \in U \subseteq A$. Often when one talks of a neighbourhood of x, they just mean an open set containing x.

O

Open Ball**

Given a metric space (X, d), a point $x \in X$ and a radius $r \in \mathbb{R}_{>0}$, the open ball about x of radius r is the subset of all points of X closer to x than r; $B_r(x) := \{y \in X | d(x, y) < r\}$.

Open Cover**

Given a topological space $(X, \mathcal{T})$ and a subset $A \subseteq X$, an open cover of A is a collection of open sets whose union contains A, i.e. $\{U_i\}_{i \in I} \subseteq \mathcal{T} : \cup_{i \in I} U_i \supseteq A$.

P

Paracompact

A topological space $(X, \mathcal{T})$ is paracompact if every open cover of X has a refinement which is locally finite, i.e. given $\{U_i | i \in I\} \subset \mathcal{T}$ such that $\cup_{i \in I} U_i = X$, there exists a collection of open sets $\{V_j | j \in J\} \subseteq \mathcal{T}$ such that $\cup_{j \in J} V_j = X, \forall j \in J : \exists i \in I : V_j \subseteq U_i$, and $\forall x \in X : \exists \mathcal{N}_x \in \mathcal{T} : x \in \mathcal{N}_x \wedge \#\{j \in J | \mathcal{N}_x \cap V_j \neq \emptyset\}$ (a refinement of an open cover is a second cover such that each open set of the refinement is contained in at least one of the sets of the original cover and is locally finite if, for each point in the space, there is some neighbourhood of the point such that a finite number of sets in the refinement intersect the neighbourhood). Paracompactness is a weaker property than compactness, and is often a required property of manifolds as every open cover of a paracompact space admits a partition of unity.

Partial Order**

A partial order $(P, \leq)$ is a set P equipped with a relation $\leq$ which is reflexive, antisymmetric, and transitive, e.g.$(\mathcal{P}(X), \subseteq)$, where X is any set and $\mathcal{P}(X)$ is its power set.

Partition**

A partition of a set X is a collection $P \subseteq \mathcal{P}(X)$ of non-empty subsets of X such that their union is all of X, $\cup P = \cup_{A \in P} A = X$ and they are pairwise disjoint: $\forall A, B \in P : A \cap B = \emptyset \Leftrightarrow A \neq B$. Equivalently, each element of x is contained in a unique element of P, $\forall x \in X : \exists! A \in P : x \in A$.

(Axiom of the) Power Set**

Given a set X, its 'power set' $\mathcal{P}(X)$ is the set of all subsets of X; $\mathcal{P}(X) = \{U | U \subseteq X\}$. The 'axiom of the power set', one of the Zermelo-Fraenkel axioms, asserts that for any set X, there exists a set whose elements are precisely the subsets of X (i.e. the power set), in symbols: $\forall X : \exists \mathcal{P}(X) : \forall U : (U \in \mathcal{P}(X) \iff \forall a : a \in U \Rightarrow a \in X)$.

Product (Category Theory)**

Given a category $\mathcal{C}$ and a collection $(A_i)_{i \in I}$ of objects in $\mathcal{C}$, a product of the A_i is an object $\prod_{i \in I} A_i$ of $\mathcal{C}$ and for each $i \in I$ an arrow $\pi_i : \prod_{j \in I} A_j \to A_i$ satisfying the following universal property: for any object X of $\mathcal{C}$ with a collection of arrows $(f_i : X \to A_i)_{i \in I}$, there exists a unique arrow $\left(\prod_{i \in I} f_i\right) : X \to \prod_{i \in I} A_i$

such that $\forall i \in I : \pi_i \circ \left(\prod_{i\in I} f_i\right) = f_i$. If a product exists, then it is unique up to unique isomorphism (if $(X, (\pi_i)_{i\in I})$ and $(Y, (\phi_i)_{i\in I})$ are both products of $(A_i)_{i\in I}$ in $\mathcal{C}$, then there exists a unique arrow $u : X \to Y$ such that $\forall i : \pi_i = \phi_i \circ u$, and in fact u must be an isomorphism. The categorical product is the prototypical example of a limit.

Product Topology**

Given two topological spaces $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, one defines the product topology on their Cartesian product $X \times Y$, $\mathcal{T}_{X\times Y}$, as the coarsest topology on $X \times Y$ to contain all sets $U \times V$ where $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$, i.e. the topological space with basis $\{U \times V \subseteq X \times Y | U \in \mathcal{T}_X \wedge V \in \mathcal{T}_Y\}$. The space $(X \times Y, \mathcal{T}_{X\times Y})$ is the product of the aforementioned spaces in the categorical sense (with projection maps $\pi_1 : X \times Y \to X, (x, y) \mapsto x,\ \ \pi_2 : X \times Y \to Y, (x, y) \mapsto y$.

Proposition**

In metamathematics, a proposition is a statement with definite truth value (in classical logic the truth values are simply 'True' and 'False', but in other models of nonstandard logic the truth values could be e.g. any element of $[0, 1]$, or of a more general Boolean algebra.

Q

Quotient Set**

Given a set X and an equivalence relation $\sim$ on X, the quotient of X by $\sim$ is the set of all equivalence classes of X under $\sim$, i.e. $X/\sim := \{[x] \subseteq X | x \in X\}$.

R

Reflexive**

A relation $\mathscr{R}$ is reflexive on a set X if each element of X is related to itself, $\forall x \in X : x\mathscr{R}x$.

Relation**

A relation is a proposition of two variables, where typically each variable may belong to a fixed set or class; for R a relation on X, Y we typically write this proposition as either $R(x, y)$ or xRy. One often identifies a relation R with its graph $\Gamma_R = \{(x, y) \in X \times Y | xRy\}$, and if Y is not specified one interprets 'R is a relation on X' as $\Gamma_R \subseteq X \times X$.

(Principle of) Restricted Comprehension**

The principle of restricted comprehension states that, for any set x and predicate P, there exists a set (written $\{a \in x | P(a)\}$) whose elements are precisely those elements a of x which satisfy $P(a)$: $\forall x : \exists y : \forall a : (a \in y \Leftrightarrow a \in x \wedge P(a))$. This is an axiom schema, as a copy of it applies for every predicate P, and it is implied by the stronger axiom of replacement: take a set x and a predicate P, then either there is no element of x satisfying P (and so the required set is $\emptyset$), or there is some $\tilde{a} \in x$ such that $P(a)$. In the second case, one forms a relation $\mathscr{R}$ by $a\mathscr{R}b \Leftrightarrow a \in x \wedge ((P(a) \wedge a = b) \vee (\neg P(a) \wedge b = \tilde{a}))$ which is functional on x, and the image of each $a \in x$ is either a itself (if $P(a)$ holds) or $\tilde{a}$ (if $P(a)$ is false), so the image of x under $\mathscr{R}$ is precisely those elements of x for which P is true. A stronger version of the principle of restricted comprehension which is not an axiom of set theory (as it leads to inconsistencies) is the principle of unrestricted comprehension, which would allow for sets of the form $\{x | P(x)\}$.

Ring**

A (unital) ring $(R, +, \times)$ consists of a set R with binary operations $+, \times : R \times R \to R$ ('addition' and 'multiplication') such that $(R, +)$ is an abelian group with identity 0_R, $(R, \times)$ is a monoid with identity 1_R, and $\times$ distributes over $+$, i.e $\forall a, b, c \in R : (a + b) \times c = a \times c + b \times c, a \times (b + c) = a \times b + a \times c$. One generally also requires that $0_R \neq 1_R$. Sometimes a ring is not required to have a multiplicative identity, so that $R, \times$ is instead a semigroup.

S

Semigroup

A semigroup $(S, \cdot)$ consists of a set S with a binary product $\cdot : S \times S \to S$ which is associative.

Semisimple Lie Algebra**

A Lie algebra L is called 'semisimple' if it has ideals $(\mathfrak{g}_i)_{i \in I}$ such that each $\mathfrak{g}_i$ is simple and L is their direct sum, $L = \bigoplus_{i \in I} \mathfrak{g}_i$.

Sequence**

Given a set X (which may additionally be a topological space or metric space or some other structure with an underlying set), a sequence in X is just a function $a : \mathbb{N} \to X, n \mapsto a_n$. It is common to denote a sequence by its terms, $(a_i)_{i=0}^{\infty}$, instead of refering to the function itself.

Set**

Loosely speaking, a set is a collection of elements, denoted $a \in X$ ('a is contained in the set X'). One may denote a set by a rule which its members obey: $X = \{a \in Y | P(a)\}$, for Y some larger set and $P(a)$ some proposition. More formally, a set is an object which obeys a set of axioms, typically the Zermelo-Fraenkel axioms and the axiom of choice, which tell us what is and is not a set.

Simple Lie Algebra**

A Lie algebra L is called 'simple' if it is not abelian and also has no ideals other than L and 0.

Span**

Given a vector space $V, +, \cdot$ over a field k, the span of a subset $M \subseteq V$, written $\text{span}_k(M)$ or $\langle M \rangle$, is the set of all finite linear combinations of elements of M, $\text{span}_k(M) = \{\sum_{r=1}^{n} a_r v_r | n \in \mathbb{Z}_{\geq 0},\ a_r \in k,\ v_r \in M\}$. The span is also the smallest (by inclusion) sub-vector space of V to contain M or, equivalently, the intersection of all subspaces which contain M. If V is further a topological vector space, then the closure of the span, $overline\text{span}_k(M)$, is the set of all convergent (possibly infinite) linear combinations of elements of M.

Spectrum (of a Commutative Ring)

The spectrum of a commutative ring R, $\text{Spec}(R)$, is the set of all its prime ideals. The spectrum may be made into a topological space by endowing it with the Zariski topology - the closed sets are given by $V_I := \{\mathfrak{p} \in \text{Spec}(R) | I \subseteq \mathfrak{p}\}$, where I is any ideal of R. In this way, Spec is a contravariant functor from **CRing** to **Set**, as any ring homomorphism $\phi : R \to S$ induces a map $\mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$ which is continuous with respect to this topology.

Subalgebra**

Given an algebra $(\mathfrak{A}, \diamond)$ over a ring R, a subalgebra of $\mathfrak{A}$ is a nonempty subset $\mathfrak{B} \subset \mathfrak{A}$ such that $(\mathfrak{B}, \diamond)$ is also an algebra, i.e. $\forall x, y \in \mathfrak{B} : \forall r \in R : x + y \in \mathfrak{B} \wedge rx \in \mathfrak{B} \wedge -x \in \mathfrak{B} \wedge x \diamond y \in \mathfrak{B}$.

Subgroup

Given a group $(G, \cdot)$, a 'subgroup' of G is a non-empty subset $H \subseteq G$ such that $(H, \cdot)$ is also a group, i.e. $\forall g, h \in H : g \cdot h \in H,\ e_G \in H,\ g^{-1} \in H$. One writes $H \leq G$, and calls H 'proper' if $H \neq G$ and 'non-trivial' if $H \neq \{e_G\}$.

Subset**

A 'subset' of a set X is a set U whose elements are all elements of X. We denote this relation by $U \subseteq X$; $U \subseteq X \Leftrightarrow \forall a : a \in U \Rightarrow a \in X$. If $U \neq X$, then we call U a 'proper' subset of X.

Subset Product (Algebra)**

Given an algebra $(\mathcal{A}, \diamond)$ over a field k and two subsets $S, T \subseteq \mathcal{A}$, their product $S \diamond T$ is the subspace of $\mathcal{A}$ spanned by all pairs $s \diamond t$ with $s \in S$, $t \in T$, i.e. $S \diamond T := \{\sum_{i=1}^{n} a_i(s_i \diamond t_i) | a_i \in k,\ s_i \in S,\ t_i \in T\}$.

Subspace Topology**

Given a topological space X with topology $\mathcal{T}$, any subset $A \subseteq X$ can be made into a topological space by the 'subspace topology' $\mathcal{T}|_A = \{A \cap U \subseteq A | U \in \mathcal{T}\}$, where the open subsets of A are the intersections of A with the open subsets of X.

Superset**

A 'superset' of a set X is a set S which contains all elements of X. We denote this relation by $S \supseteq X$; $S \supseteq X \Leftrightarrow \forall a : a \in X \Rightarrow a \in S$. If $S \neq X$, then we call S a 'proper' superset of X.

Supremum**

Given a Partial Order $(P, \leq)$ and a subset $A \subseteq P$, the supremum of A, if it exists, is the least upper bound of A, i.e. the element $\sup A \in P$ such that $\forall x \in P : (\forall a \in A : a \leq x \Rightarrow \sup A \leq x)$. If the supremum exists it must be unique.

Symmetric Relation**

A relation R on a set X is symmetric if $\forall x, y \in X : xRy \Rightarrow yRx$. A relation is antisymmetric if $\forall x, y \in X : (xRy \wedge yRx) \Rightarrow x = y$.

T

Terminal Object**

An object τ of a category $\mathcal{C}$ is called 'terminal' if, for any other object x of $\mathcal{C}$, there is a unique arrow from x to τ, i.e. $\forall x \in \mathcal{C}_0 : \exists! \tau_x \in \mathcal{C}_1 : \mathrm{dom}(\tau_x) = x \wedge \mathrm{cod}(\tau_x) = \tau$ or (in terms of hom-sets) $\forall x \in \mathcal{C}_0 : \mathrm{Hom}_{\mathcal{C}}(x, \tau) = \{\tau_x\}$. If a category has terminal objects they are all isomorphic to each other.

Topological Field**

A topological field $(k, \mathcal{T}, +, \cdot)$ is a topological space $(k, \mathcal{T})$ such that $(k, +, \cdot)$ is a field and $\cdot, + : k \times k \to k$ are both continuous when $k \times k$ is endowed with the product topology.

Topological Space**

A 'topological space' $(X, \mathcal{T}_X)$ consists of a set X (the 'space') and a set $\mathcal{T}_X \subseteq \mathcal{P}(X)$ of subsets of X, the 'topology' on X, the elements of which are called the 'open subsets' of X. $\mathcal{T}_X$ is required to satisfy the following: The total space and the empty set must be open: $X, \emptyset \in \mathcal{T}_X$; The union of any collection of open sets must be open: $\forall \mathcal{U} \subseteq \mathcal{T}_n : \cup \mathcal{U} \in \mathcal{T}_X$; The intersection of two (and hence any finite number of) open sets must be open: $\forall U_1, U_2 \in \mathcal{T}_X : U_1 \cap U_2 \in \mathcal{T}_X$.

Topological Vector Space**

A topological vector space $(V, \mathcal{T}, +, \cdot)$ over a topological field k is a k-vector space $(V, +, \cdot)$ such that $(V, \mathcal{T})$ is a topological space, and the maps $+ : V \times V \to V$ and $\cdot : k \times V \to V$ are both continuous when $V \times V$ and $k \times V$ are endowed with the product topology.

Total Order**

A partial order $(P, \leq)$ is further a total order if any two elements are comparible, i.e. $\forall x, y \in P : x \leq y \vee y \leq x$.

Transitive**

A relation R on a set X is transitive if $\forall x, y, z \in X : (xRy \wedge yRz) \Rightarrow xRz$.

U

(Axiom of) Union**

Given a set X, the union of X, denoted $\cup X$, is the set which contains the elements of all the sets in X - in symbols: $a \in \cup X \Leftrightarrow \exists x \in X : a \in x$, or equivalently $\cup X = \{a | \exists x \in X : a \in x\}$. If $X = \{x_1, ..., x_n\}$ consists of a finite number of sets then we often write the union of X as $\cup X = x_1 \cup \cdots \cup x_n$. If $X = \{x_i | i \in I\}$ is an indexed collection of sets then we often denote the union as $\cup X = \cup_{i \in I} X$. The axiom of union, one of the axioms of Zermelo-Fraenkel set theory, states that for any set X, its union is also a set: $\forall X : \exists \cup X : \forall a : (a \in \cup X \Leftrightarrow \exists x : a \in x \wedge x \in X)$.

Universal Enveloping Algebra**

Given a Lie algebra $\mathfrak{g}$ over a ring R, the universal enveloping algebra of $\mathfrak{g}$, $U(\mathfrak{g})$, is an associative R-algebra which contains $\mathfrak{g}$ as a sub algebra. It is constructed by taking the free associative R-algebra over $\mathfrak{g}$, $R\langle\mathfrak{g}\rangle = A$, and taking the quotient by the sub-module spanned by: $\{x +_A y - (x +_{\mathfrak{g}} y),\ r \cdot_A x - (r \cdot_{\mathfrak{g}} x,\ xy - yx - [x, y] | r \in R,\ x, y \in \mathfrak{g}\}$.

V

Vector Space**

Given a field $(k, +, \times, 0_k, 1_k)$, a 'vector space over k', or a 'k-vector space' $(V, +, \cdot)$ is a set V, with a binary operation $+ : V \times V \to V$ ('vector addition') such that $(V, +)$ is an abelian group, and an operation $\cdot : k \times V \to V$ ('scalar multiplication') which is left-unital (i.e. $\forall v \in V : 1_k \cdot v = v$), associates with multiplication in the field (i.e. $\forall a, b \in k : \forall v \in V : a \cdot (b \cdot v) = (a \times b) \cdot v$), and distributes over both scalar and vector addition ($\forall a, b \in k : \forall u, v \in V : (a + b) \cdot (u + v) = a \cdot u + a \cdot v + b \cdot u + b \cdot v$).

Vertical Composition**

Given three parallel functors $F, G, H : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$, $\mathcal{D}$ and two natural transformations $\eta : F \Rightarrow G$, $\xi : G \Rightarrow H$, their vertical composition is the natural transformation $(\xi \circ \eta) : F \Rightarrow H$ defined by $(\xi \circ \eta)_x = (\xi_x \circ \eta_x) : F(x) \to H(x)$.

W

X

Y

Z

Zermelo-Fraenkel Axioms of Set Theory**

The Zermelo-Fraenkel axioms of set theory (usually along with the axiom of choice) are a collection of 8 axioms on the nature of sets. The axioms are those of: extensionality, the empty set, the pair set, unions, replacement, power set, infinity, and foundation.

Zorn's Lemma**

Given any partial order $(P, \leq)$ such that every chain in P has an upper bound, P has a maximal element. This is equivalent to the axiom of choice and the statement ''every vector space has a basis".