

CATEGORY THEORY

2-Category

SAM'S MATHS DICTIONARY

GROUP THEORY

*Abelian Group***

SAM'S MATHS DICTIONARY

LIE ALGEBRAS

*Abelian Subalgebra***

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A 2-Category \mathcal{C} is a **category** which is **enriched** over **Cat**, i.e for each pair of objects $x, y \in \mathcal{C}_0$, the **hom-set** $\mathcal{C}(x, y)$ forms a category, i.e. there is a collection \mathcal{C}_2 of 2-arrows, each of which has a domain and codomain a pair of parallel 1-arrows in \mathcal{C}_1 , and these behave nicely under composition of 1-arrows. In particular, for 1-arrows $f, g : X \rightarrow Y$ and $h, k : Y \rightarrow Z$, with 2-arrows $\eta : f \Rightarrow g$ and $\xi : h \Rightarrow k$, there is a composite 2-arrow

$$(\xi * \eta) : (h \circ f) \Rightarrow (k \circ g).$$

A **group** (G, \cdot) is ‘abelian’ if the group operation is **commutative**, i.e.

$$\forall g, h \in G : g \cdot h = h \cdot g.$$

Given a **Lie algebra** \mathfrak{g} , a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is called abelian if it has a trivial bracket with itself: $\forall x, y \in \mathfrak{h} : [x, y] = 0$. Equivalently, the **subalgebra product** of \mathfrak{h} with itself is trivial, $[\mathfrak{h}, \mathfrak{h}] = 0$. If all of \mathfrak{g} is an abelian subalgebra, we call \mathfrak{g} abelian.

ALGEBRA

*Algebra***

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LINEAR ALGEBRA

*Alternating***

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SET THEORY

*Associative***

SAM'S MATHS DICTIONARY

An algebra (\mathcal{A}, \diamond) over a ring R is an R -module together with an R -bilinear binary product $\diamond : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ - i.e. for all $x, y, z, w \in \mathcal{A} : \forall r, s \in R : (r \cdot x + y) \diamond (s \cdot z + w) = (rs) \cdot (x \diamond z) + r \cdot (x \diamond w) + s \cdot (y \diamond z) + y \diamond w$. Typically one specifies the case where R is a field and \diamond is associative.

Given an abelian group $(A, +)$ and a binary product $[\cdot, \cdot] : A \times A \rightarrow A$ (typically $(A, +)$ has the additional structure of being a vector space and $[\cdot, \cdot]$ is also bilinear, but this is not required), one says that $[\cdot, \cdot]$ is alternating if $\forall x \in A : [x, x] = 0$.

Given a binary product $\cdot : X \times X \rightarrow X$ on a set X , one says that \cdot is associative if the order of calculation does not matter for repeated applications of \cdot , i.e. $\forall x, y, z \in X : x \cdot (y \cdot z) = (x \cdot y) \cdot z$. More generally if \circ is a partial function on X , \circ is called associative if $f \circ (g \circ h) = (f \circ g) \circ h$ whenever $f, g, h \in X$ such that $f \circ g, g \circ h, f \circ (g \circ h), (f \circ g) \circ h$ exist, for example composition of arrows in a category. Finally, given multiple sets and functions between them, $\cdot : A \times A \rightarrow A$ and $\star : A \times B \rightarrow B$, one says that \cdot, \star associate with each other if $\forall \alpha, \beta \in A : \forall x \in B : \alpha \star (\beta \star x) = (\alpha \cdot \beta) \star x$, for example multiplication in the field and scalar multiplication for a vector space.

SET THEORY

*Axiom of Choice***

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SET THEORY

*Axiom of Extensionality***

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SET THEORY

*Axiom of Foundation***

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An axiom of **set** theory, typically included alongside the **Zermelo-Fraenkel axioms** (from which it is independent) as one of the foundational axioms of set theory. Its statement is “For any collection of **nonempty**, **disjoint**, there exists a set which contains precisely each element from each member of the collection.” In symbols:

$\forall x : (\emptyset \notin x \wedge \forall a, b \in x : a \cap b = \emptyset) \Rightarrow \exists c : \forall a \in x : \exists! y \in a : y \in c$. The axiom of choice is equivalent to **Zorn’s Lemma**, and statements such as “Every **set** has a **total order**”, “every **vector space** has a **basis**”.

One of the **Zermelo-Fraenkel** axioms of **set theory**, which states that two sets are equal precisely if they share all their elements:

$\forall x : \forall y : x = y \Leftrightarrow (\forall a : a \in x \Leftrightarrow a \in y)$. This axiom means that, for example, $\{x, y\} = \{y, x\} = \{x, x, y\}$ i.e. ordering and repeats do not affect the definition of a set.

One of the **Zermelo-Fraenkel** axioms of **set** theory, which states that every **non-empty** set x contains an element which is **disjoint** from x :

$\forall x : x \neq \emptyset \Rightarrow (\exists a \in x : a \cap x = \emptyset)$. Important consequences are that no set may contain itself, nor can there exist ‘cycles’ in the epsilon relation (i.e. a sequence of sets x_1, \dots, x_n satisfying $x_1 \in x_2 \in \dots \in x_n \in x_1$, nor can there exist a chain of ‘infinite descent’ (an infinite sequence x_1, x_2, \dots such that $x_1 \ni x_2 \ni \dots$).

SET THEORY

*Axiom of Infinity***

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SET THEORY

*Axiom of Replacement***

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SET THEORY

*Axiom of the Pair Set***

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One of the **Zermelo-Fraenkel** axioms of **set theory**, which asserts the existence of an infinite set, which has the same **cardinality** as the natural numbers. $\exists I : (\emptyset \in I \wedge \forall x : x \in I \Rightarrow x \cup \{x\} \in I$. As such, I contains $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \dots$ and so on.

One of the **Zermelo-Fraenkel** axioms of **set** theory, which states that whenever x is a set and \mathcal{R} is a **functional relation** on x , the image of x under \mathcal{R} (the set of all b such that $a\mathcal{R}b$ for some $a \in x$) is a set:
 $\forall x : [\forall a \in x : \exists! b : a\mathcal{R}b] \Rightarrow [\exists y : \forall b : b \in y \Leftrightarrow \exists a \in x : a\mathcal{R}b]$. This is actually an axiom schema (as it holds for every relation \mathcal{R}) and underlies the **principle of restricted comprehension**.

An axiom of Zermelo-Fraenkel set theory which states that, for any two sets X, Y , there exists a set $(\{X, Y\})$ whose elements are precisely X and Y :
 $\forall X : \forall Y : \exists Z : \forall a : (a \in Z \Leftrightarrow a = X \vee a = Y)$.

FUNCTIONAL ANALYSIS

Banach Space

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LINEAR ALGEBRA

*Basis (Linear Algebra)***

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SET THEORY

*Binary Product***

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A Banach space $(\mathcal{B}, \|\cdot\|)$ is a **normed vector space** which is **Cauchy complete** with respect to the **metric** induced by the norm: $d(x, y) = \|x - y\|$.

A basis of a **vector space** $(V, +, \cdot)$ over a **field** k is a **subset** $S \subseteq V$ which is **linearly independent** and **spans** V . In infinite dimensions, one specifies a difference between a ‘Hamel basis’ (using finite sums for linear independence and span) and a ‘Schauder basis’ (using convergent infinite sums for linear independence and span).

Given a set X , a binary product (or ‘binary operation’) on X is a function $\cdot : X \times X \rightarrow X$, typically written $\cdot : (x, y) \mapsto \cdot(x, y) = x \cdot y$.

ORDER THEORY

*Bound (Order Theory)***

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METRIC SPACES

*Bounded Set***

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DIFFERENTIAL GEOMETRY

*Bundle (Topology)***

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Given a **partial order** (P, \leq) and a subset $S \subseteq P$, an upper (resp. lower) bound of S is an element $x \in P$ such that $\forall s \in S : s \leq x$ (resp. $x \leq s$).

Given a **metric space** (X, d) , a **subset** $A \subseteq X$ is called ‘bounded’ if it has a finite **diameter**, i.e. $\exists r \in \mathbb{R} : \forall x, y \in A : d(x, y) \leq r$.

A ‘bundle’ (E, p, B) consists of **sets** (or, usually, **topological spaces**) E (the ‘total space’) and B (the ‘base space’), and a map (or a **continuous map**, for topological spaces) $p : E \rightarrow B$.

SET THEORY

*Cartesian Product***

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CATEGORY THEORY

*Category***

SAM'S MATHS DICTIONARY

METRIC SPACES

*Cauchy Complete***

SAM'S MATHS DICTIONARY

Given **sets** X, Y , their Cartesian product $X \times Y$ consists of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$ (by ‘ordered pair’ one means that $(x, y) = (x', y') \Leftrightarrow (x = x' \wedge y = y')$). This set is constructible within the **Zermelo-Fraenkel axioms** of set theory by identifying the set $\{\{x\}, \{x, y\}\}$ with the pair (x, y) , so that $X \times Y = \{S \in \mathcal{P}(\mathcal{P}(X \cup Y)) \mid \exists x \in X : \exists y \in Y : S = \{\{x\}, \{x, y\}\}\}$.

A category $C = (C_0, C_1, \text{dom}, \text{cod}, 1, \circ)$ consists of a collection (formally a member of some **Grothendieck universe**) C_0 of ‘objects’, a collection C_1 of ‘arrows’ (or ‘morphisms’), functions $\text{dom}, \text{cod} : C_1 \rightarrow C_0$ which assign to each arrow its domain (source) and codomain (target) respectively, a function $1 : C_0 \rightarrow C_1, X \mapsto 1_X$ which assigns to each object an ‘identity arrow’, and a partial function $\circ : C_1 \times C_1 \rightarrow C_1, (f, g) \mapsto f \circ g$ so that $f \circ g$ is defined precisely when $\text{dom}(f) = \text{cod}(g)$, ‘function composition’. The identity map must satisfy $\forall X \in C_0 : \text{dom}(1_X) = \text{cod}(1_X) = X$, and the composition (where it is defined) must satisfy $\text{dom}(f \circ g) = \text{dom}(g)$ and $\text{cod}(f \circ g) = \text{cod}(f)$. Composition must be **associative**, so that $\forall f, g, h \in C_1 : (\text{dom}(f) = \text{cod}(g) \wedge \text{dom}(g) = \text{cod}(h)) \Rightarrow f \circ (g \circ h) = (f \circ g) \circ h$, and **unital** via the identity, i.e. $\forall X \in C_0 : \forall f \in C_1 : (\text{dom}(f) = X \Rightarrow f \circ 1_X = f) \wedge (\text{cod}(f) = X \Rightarrow 1_X \circ f = f)$.

A **metric space** metric space (X, d) is ‘Cauchy complete’ if every **Cauchy sequence** in X has a **limit** in X . For example: \mathbb{R} is Cauchy complete by definition, while \mathbb{Q} is not, as the sequence $\sum_{r=0}^n \frac{1}{n!}$ is Cauchy but its limit, e , is not in \mathbb{Q} . Given a metric space (X, d) , its Cauchy (or metric) completion (\bar{X}, \bar{d}) is the **quotient** of the set of all Cauchy sequences in X modulo the **equivalence relation** $(a_i)_{i=0}^\infty \sim (b_i)_{i=0}^\infty : \Leftrightarrow \lim_{i \rightarrow \infty} d(a_i, b_i) = 0$, so that two sequences are equivalent if they share a limit point, with an induced metric $\bar{d}([(a_i)_{i=0}^\infty], [(b_i)_{i=0}^\infty]) := \lim_{i \rightarrow \infty} d(a_i, b_i)$. \mathbb{R} is the metric completion of \mathbb{Q} .

METRIC SPACES

*Cauchy Sequence***

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ORDER THEORY

*Chain (Order Theory)***

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HOMOLOGICAL ALGEBRA

*Chain Complex***

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Given a **metric space** (X, d) , a **sequence** $(a_i)_{i=0}^{\infty}$ is ‘Cauchy’ if the **diameter** of the partial sequence $(a_i)_{i=n}^{\infty}$ tends to 0 as $n \rightarrow \infty$, i.e.
 $\forall \epsilon > 0 : \exists N \geq 0 : \forall n, m \geq N : d(a_n, a_m) < \epsilon$ - for any given radius, there exists a point in the sequence beyond which all points are within that radius of each other.

A chain of a **partial order** (P, \leq) is a **subset** $S \subseteq P$ such that (S, \leq) is a **total order**.

Given an **abelian category** \mathcal{A} , a chain complex with coefficients in \mathcal{A} (C_{\bullet}) is a sequence $(C_n)_{n \in \mathbb{Z}}$ of objects of \mathcal{A} (the ‘ n -chains’ of C_{\bullet}) with a collection of maps $d_n : C_n \rightarrow C_{n-1}$ (the ‘differential’ of C_{\bullet}) such that $d_{n-1} \circ d_n = 0$ for each $n \in \mathbb{Z}$. The **kernels** and **images** of the differential, $Z_n = \ker(d_n) \subseteq C_n$ and $B_n = \ker(d_{n+1}) \subseteq C_n$ are called the n -cycles and n -boundaries respectively, and the **quotient** $H_n = Z_n/B_n$ is called the n^{th} homology object of C_{\bullet} .

SET THEORY

*Class***

SAM'S MATHS DICTIONARY

METRIC SPACES

*Closed Ball***

SAM'S MATHS DICTIONARY

TOPOLOGY

*Closed Set***

SAM'S MATHS DICTIONARY

Within **Zermelo-Fraenkel** set theory (and other axiomatisations), a class is a collection of **sets** satisfying some predicate, for example $\{x|R(x)\}$, where R is a predicate which identifies x as describing a **ring**. Classes are often not sets as they are 'too large', in which case the class is called 'proper' - the aforementioned collection of all rings is an example of a proper class.

Given a **metric space** (X, d) , a point $x \in X$ and a radius $r \in \mathbb{R}_{>0}$, the closed ball about x of radius r is the **subset** of all points of X at a distance at most r from x ; $\bar{B}_r(x) := \{y \in X | d(x, y) \leq r\}$.

Given a **topological space** (X, \mathcal{T}) , a **subset** $A \subseteq X$ is 'closed' (with respect to the topology \mathcal{T}) if its **complement** is open, i.e. $X \setminus A \in \mathcal{T}$. The definition of a topology means that \emptyset and X are always closed, as is the **union** of a finite number of closed sets and the **intersection** of an arbitrary number of closed sets.

TOPOLOGY

*Coarse/Fine (Topology)***

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CATEGORY THEORY

*Co-Cone Category***

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HOMOLOGICAL ALGEBRA

*Cokernel***

SAM'S MATHS DICTIONARY

Given a **set** X and two **topologies** $\mathcal{T}, \mathcal{T}'$ on X , one says that \mathcal{T} is coarser than \mathcal{T}' (or, equivalently, \mathcal{T}' is finer than \mathcal{T}) if $\mathcal{T} \subseteq \mathcal{T}'$. This turns the set of all topologies on X into a **partial order**. One often seeks the coarsest topology which satisfies a given condition, for example the **product topology** is the coarsest topology on $X \times Y$ which makes both $\pi_1 : X \times Y \rightarrow X, (x, y) \mapsto x$ and $\pi_2 : X \times Y \rightarrow Y, (x, y) \mapsto y$ **continuous**. The coarsest topology on any set X is $\{\emptyset, X\}$ (the ‘trivial’ or ‘indiscrete’ topology), while the finest is $\mathcal{P}(X)$ (the ‘discrete’ topology).

Given a (typically **small**) **category** \mathcal{I} , a category \mathcal{C} , and a **functor** $F : \mathcal{I} \rightarrow \mathcal{C}$, a co-cone under F is an object $X \in \mathcal{C}_0$ with a collection of arrows to X from the image of each object of \mathcal{I} under F , $\{(f_i : F(i) \rightarrow X) \in \mathcal{C}_1 \mid i \in \mathcal{I}_0\}$, which is compatible with arrows in \mathcal{I} i.e. $\forall (g : i \rightarrow j) \in \mathcal{I}_1 : f_i = F_j \circ F(g)$. Given two co-cones $(X, \{f_i\})$ and $(Y, \{g_i\})$, an arrow between them is some $(u : X \rightarrow Y) \in \mathcal{C}_0$ which respects the mappings from F , i.e. $\forall i \in \mathcal{I}_0 : f_i = u \circ g_i$. The collection of all co-cones over F with arrows between them thus forms a category, $(F \downarrow \Delta)$, an example of a **comma category**.

Given an arrow $f : X \rightarrow Y$ in an **additive category** \mathcal{A} , its cokernel is its **coequaliser** with the zero map $0 : X \rightarrow Y$, i.e. an object $\text{coker}(f) \in \mathcal{A}_0$ together with an arrow $\text{coker}(f) : Y \rightarrow \text{coker}(f)$ satisfying the **universal property** that, for $g : Y \rightarrow Z$ any arrow in \mathcal{A} such that $g \circ f = 0$, there is a unique arrow $\bar{g} : \text{coker}(f) \rightarrow Z$ such that $g = \bar{g} \circ \text{coker}(f)$, i.e. any map which vanishes upon precomposition with f factors through the cokernel. In the category of **abelian groups**, for example, the cokernel of a map is the **quotient** of the codomain by the **image** (the arrow is simply the canonical projection $Y \rightarrow Y/\text{im}f$).

SET THEORY

*Commutative***

SAM'S MATHS DICTIONARY

ALGEBRA

*Commutative Ring***

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LIE ALGEBRAS

*Commutator Lie Algebra***

SAM'S MATHS DICTIONARY

A **binary operation** $*$ on a **set** X is called commutative if the order of arguments is irrelevant, i.e. $\forall x, y \in X : x * y = y * x$.

A **ring** $(R, +, \times)$ is 'commutative' if its multiplication is **commutative**, i.e. $\forall a, b \in R : a \times b = b \times a$.

Given an **associative algebra** \mathfrak{A}, \diamond over a ring R , the commutator Lie algebra of \mathfrak{A} is the **Lie algebra** $\mathfrak{A}, [\cdot, \cdot]_{\diamond}$ with bracket $[x, y]_{\diamond} := x \diamond y - y \diamond x$. The mapping of an associative algebra to its commutator algebra is **functorial** and is **right adjoint** to the functor associating an R -Lie algebra to its **universal enveloping algebra**.

TOPOLOGY

*Compact (Topology)***

SAM'S MATHS DICTIONARY

SET THEORY

*Complement***

SAM'S MATHS DICTIONARY

CATEGORY THEORY

*Cone Category***

SAM'S MATHS DICTIONARY

A **topological space** X is ‘compact’ if every **open cover** $(U)_{i \in I}$ of X admits a finite **subcover** $(U_{i_n})_{n=1}^N$. A **subset** A of X is a compact subset if A is compact under the **subspace topology**. In a Heine-Borel **metric space** (such as, but not limited to, \mathbb{R}^n), a subset is compact iff it is **closed** and **bounded**.

Given a **subset** $A \subseteq X$ of a **set** X , the complement of A in X is the set of all elements of X which are not elements of A , i.e. $X \setminus A := \{x \in X \mid x \notin A\}$.

Given a (typically **small**) **category** \mathcal{I} , a category \mathcal{C} , and a **functor** $F : \mathcal{I} \rightarrow \mathcal{C}$, a cone over F is an object $X \in \mathcal{C}_0$ with a collection of arrows from X to the image of each object of \mathcal{I} under F , $\{(f_i : X \rightarrow F(i)) \in \mathcal{C}_1 \mid i \in \mathcal{I}_0\}$, which is compatible with arrows in \mathcal{I} i.e. $\forall (g : i \rightarrow j) \in \mathcal{I}_1 : f_j = F(g) \circ f_i$. Given two cones $(X, \{f_i\})$ and $(Y, \{g_i\})$, an arrow between them is some $(u : X \rightarrow Y) \in \mathcal{C}_0$ which respects the mappings into F , i.e. $\forall i \in \mathcal{I}_0 : f_i = g_i \circ u$. The collection of all cones over F with arrows between them thus forms a category, $(\Delta \downarrow F)$, an example of a **comma category**.

TOPOLOGY

*Continuous (Topology)***

SAM'S MATHS DICTIONARY

METRIC SPACES

*Diameter (Metric Space)***

SAM'S MATHS DICTIONARY

DIFFERENTIAL GEOMETRY

Differentiable Atlas

SAM'S MATHS DICTIONARY

Given **topological spaces** (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) , a **function** $f : X \rightarrow Y$ is called 'continuous' if whenever $U \in \mathcal{T}_Y$ is open, so is its **preimage** $f^{-1}(U) \subseteq X$, i.e.

$$\forall U \in \mathcal{T}_Y : f^{-1}(U) \in \mathcal{T}_X.$$

Given a **metric space** (X, d) , the diameter of a **subset** $A \subseteq X$ is the **supremum** of distances between pairs of points in A , i.e.

$$\text{diam}(A) = \sup_{x, y \in A} d(x, y).$$

A differentiable atlas of degree k of a **manifold** (X, \mathcal{T}) is a collection of ordered pairs ('charts') $\mathcal{A} = (U_i, \phi_i)_{i \in I}$ where $U_i \in \mathcal{T}$ and $\phi_i : U_i \rightarrow \mathbb{R}^n$ is a **homeomorphism** onto its image $\phi_i(U_i) \in \mathcal{T}_{\mathbb{R}^n}$ (where $\mathcal{T}_{\mathbb{R}^n}$ is the **metric topology** on \mathbb{R}^n). The charts must also be C^k compatible, i.e. for every pair of charts (U, ϕ) and (V, ψ) , the transition map $\psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V)$ must be **C^k differentiable**. The set of all degree- k atlases on X form a **partial order**, and a **maximal element** of this partial order (i.e. an atlas \mathcal{A} such that no other chart on X is compatible with every chart of \mathcal{A}) is called a C^k -differentiable structure on X .

SET THEORY

*Disjoint***

SAM'S MATHS DICTIONARY

SET THEORY

*(Axiom of the) Empty Set***

SAM'S MATHS DICTIONARY

CATEGORY THEORY

*Equaliser***

SAM'S MATHS DICTIONARY

Two sets A, B are disjoint if their **intersection** is **empty**, $A \cap B = \emptyset$.

The empty set \emptyset is a **set** which contains no elements; $\forall x : x \notin \emptyset$. The axiom of the empty set, one of the axioms of **Zermelo-Fraenkel** set theory, asserts the existence of such a set. By the **axiom of extensionality** the empty set is unique, and by definition is a **subset** of every set.

Given a pair of parallel arrows $f, g : X \rightarrow Y$ in a **category** \mathcal{C} , their equaliser is an object $\text{Eq}(f, g) \in \mathcal{C}_0$ with an arrow $u : \text{Eq}(f, g) \rightarrow X$ such that $f \circ u = g \circ u$ and satisfying the **universal property** that, whenever there is an arrow $h : Z \rightarrow X$ such that $f \circ h = g \circ h$, there is a unique arrow $\bar{h} : Z \rightarrow \text{Eq}(f, g)$ such that $h = u \circ \bar{h}$ - any map which makes f, g equal upon precomposition factors through the equaliser. This is an example of a **limit** in category theory. In **set**, the equaliser of two maps $f, g : X \rightarrow Y$ is the **subset** $\text{Eq}(f, g) = \{x \in X \mid f(x) = g(x)\}$ and the arrow u is just its inclusion into X .

SET THEORY

*Equivalence Relation***

SAM'S MATHS DICTIONARY

TOPOLOGY

*Fiber Bundle***

SAM'S MATHS DICTIONARY

ALGEBRA

*Field***

SAM'S MATHS DICTIONARY

A relation \sim is called an equivalence relation on a set X if it is reflexive, symmetric and transitive, i.e. $\forall x \in X : x \sim x$, $\forall x, y \in X : x \sim y \Leftrightarrow y \sim x$, and $\forall x, y, z \in X : (x \sim y \wedge y \sim z) \Rightarrow x \sim z$. Given an element $x \in X$, its ‘equivalence class’ is the subset $[x] := \{y \in X | x \sim y\}$, sometimes denoted $[x]_{\sim}$. The equivalence classes of elements of X under \sim form a partition of X .

A fiber bundle (E, B, p, F) is a bundle of topological spaces (E, B, p) along with a third topological space F (the ‘fiber’) such that $\forall x \in B : p^{-1}(x) \cong F$, i.e. the preimage of each point of the base-space under projection is homeomorphic to the fiber, and also for each $x \in B$, $\exists U \in \mathbf{Top}(B) : \exists(\phi : p^{-1}(U) \rightarrow U \times F) \in \mathbf{Top} : p = \pi_U \circ \phi$ - there is an open neighbourhood U of every point x of the base space, upon the preimage of which the projection factors through the projection onto U from the trivial product $U \times F$.

A field is a ring with commutative and invertible multiplication; The field $(k, +, \times)$ consists of a set k and binary operations $+, \times : k \times k \rightarrow k$ which are both commutative, associative, and unital (with units $0_k, 1_k$ respectively), such that $\forall x \in k : \exists(-x) \in k : x + (-x) = 0_k$, $\forall x \in k \setminus \{0_k\} : \exists x^{-1} \in k : x \times x^{-1} = 1_k$, and \times distributes over $+$, i.e. $\forall a, b, c \in k : a \times (b + c) = (a \times b) + (a \times c)$. Typically it is also required that $0_k \neq 1_k$.

FUNCTIONAL ANALYSIS

*Fréchet Space (Functional Analysis)***

SAM'S MATHS DICTIONARY

SET THEORY

*Function***

SAM'S MATHS DICTIONARY

SET THEORY

*Functional Relation***

SAM'S MATHS DICTIONARY

A Fréchet space is a **topological vector space** $(V, \mathcal{T}, +, \cdot)$ such that the **topological space** (V, \mathcal{T}) is **metrizable** (and so also **Hausdorff**), V is **Cauchy complete** with respect to this metric, and V is **locally convex**.

Given **sets** X, Y , a function $f : X \rightarrow Y$ is an assignment of precisely one element $f(x)$ of Y to each element x of X , written $x \mapsto f(x)$. This is equivalent to there being a **functional relation** \mathcal{R} on X such that the image lies within Y ; $\forall x \in X : \exists! y \in Y : x \mathcal{R} y$, with $y = f(x) \Leftrightarrow x \mathcal{R} y$.

A **relation** \mathcal{R} is called 'functional' on a **set** x if for each element $a \in x$ there is a unique b such that $a \mathcal{R} b$, i.e. $\forall a \in x : \exists! b : a \mathcal{R} b$.

CATEGORY THEORY

*Functor***

SAM'S MATHS DICTIONARY

SET THEORY

*Graph (of a function or relation)***

SAM'S MATHS DICTIONARY

GROUP THEORY

*Group***

SAM'S MATHS DICTIONARY

Given **categories** \mathcal{C} , \mathcal{D} , a (covariant) functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a pair of **maps** (typically both simply labelled F) $F_0 : \mathcal{C}_0 \rightarrow \mathcal{D}_0$ and $F_1 : \mathcal{C}_1 \rightarrow \mathcal{D}_1$ which are compatible with identities and compositions, i.e. $\forall X \in \mathcal{C}_0 : F_1(1_X) = 1_{F_0(X)}$ and $\forall X \xrightarrow{f} Y \xrightarrow{g} Z \in \mathcal{C} : F_1(g \circ f) = F_1(g) \circ F_1(f)$.

Given a **function** $f : X \rightarrow Y$ between **sets** X , Y , its graph $\Gamma_f \subseteq X \times Y$ is the **subset** of their **Cartesian product** given by all pairs $(x, f(x))$, i.e. $\Gamma_f := \{(x, y) \in X \times Y \mid x \wedge y = f(x)\}$. More generally, given a **relation** \mathcal{R} , its graph is $\Gamma_{\mathcal{R}} := \{(x, y) \mid x \mathcal{R} y\}$. If \mathcal{R} is **functional** or is a relation between sets, then $\Gamma_{\mathcal{R}}$ is a set, but in general it may be a **proper class**.

A group (G, \cdot) is a **set** G equipped with a **binary product** $\cdot : G \times G \rightarrow G$ which is **associative**, has a **unit** $e \in G$, and **has inverses**.

TOPOLOGY

*Hausdorff Space***

SAM'S MATHS DICTIONARY

CATEGORY THEORY

*Hom Set***

SAM'S MATHS DICTIONARY

TOPOLOGY

Homeomorphism

SAM'S MATHS DICTIONARY

A **topological space** (X, \mathcal{T}) is called Hausdorff if any two distinct points are contained in **disjoint neighbourhoods**;

$$\forall x, y \in X : x \neq y \Rightarrow (\exists U, V \in \mathcal{T} : x \in U \wedge y \in V \wedge U \cap V = \emptyset).$$

Given a **category** \mathcal{C} and a pair of objects $x, y \in \mathcal{C}_0$, the 'hom set' $\mathcal{C}(x, y) = \text{Hom}_{\mathcal{C}}(x, y)$ is the collection of all arrows from x to y , i.e. $\{f \in \mathcal{C}_1 \mid \text{dom}(f) = x \wedge \text{cod}(f) = y\}$.

Given **topological spaces** (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) , a **function** $f : X \rightarrow Y$ is called a 'homeomorphism' if f is **bijective** and **continuous**, and its **inverse** $f^{-1} : Y \rightarrow X$ is also continuous. If such a map exists then one writes $X \cong_{\text{Top}} Y$ and calls X and Y 'homeomorphic', the topological word for **isomorphic**. One says that $f : X \rightarrow Y$ is 'homeomorphic onto its image' if $\tilde{f} : X \rightarrow \text{im}(f)$ is a homeomorphism when $\text{im}(f)$ is endowed with the **subspace topology**.

CATEGORY THEORY

*Horizontal Composition***

SAM'S MATHS DICTIONARY

LIE ALGEBRAS

*Ideal (Lie Algebra)***

SAM'S MATHS DICTIONARY

ALGEBRA

*Ideal (Ring Theory)***

SAM'S MATHS DICTIONARY

Given three **categories** $\mathcal{A}, \mathcal{B}, \mathcal{C}$, pairs of parallel **functors** $F, G : \mathcal{A} \rightarrow \mathcal{B}$ and $H, K : \mathcal{B} \rightarrow \mathcal{C}$ and two **natural transformations** $\eta : F \Rightarrow G, \xi : H \Rightarrow K$, their horizontal composition is the natural transformation $(\xi * \eta) : H \circ F \Rightarrow K \circ G$ given by $(\xi * \eta)_x = \xi_{G(x)} \circ H(\eta_x) : HF(x) \rightarrow KG(x)$.

Given a **Lie algebra** \mathfrak{g} , a **subalgebra** $\mathfrak{h} \subseteq \mathfrak{g}$ is called an ideal if $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$, where $[\mathfrak{h}, \mathfrak{g}]$ is the standard **product of subalgebras**. The subalgebras $\mathfrak{g}, \{0\}$ are ideals for any Lie algebra \mathfrak{g} and so are called ‘trivial’.

Given a **ring** $(R, +, \times)$, a **subset** $I \subseteq R$ such that $(I, +)$ is a **subgroup** of $(R, +)$ and $RI = \{ra \mid r \in R, a \in I\} \subseteq I$ (resp. $IR = \{ar \mid r \in R, a \in I\} \subseteq I$) is called a left (resp. right) ideal of R . If I is both a left and right ideal then we call it a two-sided ideal and write $I \triangleleft R$. An ideal is called ‘proper’ if it is not R itself, and ‘nontrivial’ if it is not the zero ideal $0 = \{0_R\}$.

HOMOLOGICAL ALGEBRA

*Image (Homological Algebra)***

SAM'S MATHS DICTIONARY

ORDER THEORY

*Infimum***

SAM'S MATHS DICTIONARY

CATEGORY THEORY

*Initial Object***

SAM'S MATHS DICTIONARY

Given an arrow $f : X \rightarrow Y$ in an **additive category** \mathcal{A} , its image is the **kernel** of its **cokernel**, while its coimage is the cokernel of its kernel - in an **abelian category** these two objects are **isomorphic**.

Given a **Partial Order** (P, \leq) and a subset $A \subseteq P$, the infimum of A , if it exists, is the greatest lower bound in P of A , i.e. the element $\inf A \in P$ such that $\forall x \in P : (\forall a \in A : x \leq a \Rightarrow x \leq \inf A)$. If the infimum exists it must be unique.

An object ι of a **category** \mathcal{C} is called 'initial' if, for any other object x of \mathcal{C} , there is a unique arrow from ι to x , i.e.
 $\forall x \in \mathcal{C}_0 : \exists! \iota_x \in \mathcal{C}_1 : \text{dom}(\iota_x) = \iota \wedge \text{cod}(\iota_x) = x$ or (in terms of **hom-sets**)
 $\forall x \in \mathcal{C}_0 : \text{Hom}_{\mathcal{C}}(\iota, x) = \{\iota_x\}$. If a category has initial objects they are all **isomorphic** to each other.

SET THEORY

*Intersection***

SAM'S MATHS DICTIONARY

CATEGORY THEORY

*Isomorphism***

SAM'S MATHS DICTIONARY

LIE ALGEBRAS

*Jacobi Identity***

SAM'S MATHS DICTIONARY

Given a **set** X , its intersection $\cap X$ is the set which contains all elements contained in all sets in X : $\forall a : a \in \cap X \iff \forall x \in X : a \in x$, or equivalently $\cap X = \{a | \forall x \in X : a \in x\}$. When $X = \{x_1, \dots, x_n\}$ is finite, we typically write $\cap X = x_1 \cap \dots \cap x_n$, and when $X = \{x_i | i \in I\}$ is an indexed collection of sets we write $\cap X = \cap_{i \in I} x_i$. The existence of the intersection of X is guaranteed by the axiom of **replacement**.

Given a **category** \mathcal{C} , an arrow $f : X \rightarrow Y$ in \mathcal{C} is called an isomorphism (or ‘invertible’) if there exists an arrow $\tilde{f} : Y \rightarrow X$ such that $f \circ \tilde{f} = 1_Y$ and $\tilde{f} \circ f = 1_X$. If such an \tilde{f} exists then it is necessarily unique and generally written f^{-1} , and X and Y are called ‘isomorphic’. Isomorphism is an **equivalence relation** on the objects of \mathcal{C} , and means that X and Y are essentially the same (from the point of view of the category).

Given an **abelian group** $(A, +)$ and a **binary product** $[\cdot, \cdot] : A \times A \rightarrow A$ (typically $(A, +)$ has the additional structure of being a **vector space** and $[\cdot, \cdot]$ is also **bilinear**, but this is not required), one says that $[\cdot, \cdot]$ obeys the Jacobi identity if $\forall x, y, z \in A : [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

HOMOLOGICAL ALGEBRA

*Kernel***

SAM'S MATHS DICTIONARY

LIE ALGEBRAS

*Lie Algebra***

SAM'S MATHS DICTIONARY

CATEGORY THEORY

*Limit (Category Theory)***

SAM'S MATHS DICTIONARY

In an **additive category** \mathcal{A} , the ‘kernel’ of an arrow $f : X \rightarrow Y$ refers to both an object $\ker(f) \in \mathcal{A}_0$ and an arrow $(\ker(f) : \ker(f) \rightarrow X) \in \mathcal{A}_1$. The kernel of f is the **equaliser** between f and the **0-map** from X to Y , the equaliser being a specific example of a **limit**: given any object $Z \in \mathcal{A}_0$ and any arrow $(g : Z \rightarrow X) \in \mathcal{A}_1$ such that $f \circ g = 0 \circ g = 0$, there exists a unique arrow $\tilde{g} : Z \rightarrow \ker(f)$ such that $g = \ker(f) \circ \tilde{g}$ - any map which vanishes upon composition with f factors through the kernel of f .

A Lie algebra over a **ring** R is a **left R -module** \mathfrak{g} equipped with an **R -bilinear** map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ which is **alternating** ($\forall x \in \mathfrak{g} : [x, x] = 0$) and satisfies the **Jacobi Identity** ($\forall x, y, z \in \mathfrak{g} : [x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$).

Given a (typically **small**) **category** \mathcal{I} , a category \mathcal{C} , and a **functor** $F : \mathcal{I} \rightarrow \mathcal{C}$, a limit of F is an **initial object** in the category of **cones** over F , i.e. an object $\lim_{\mathcal{I}} F \in \mathcal{C}_0$ and a collection of arrows $\{(\phi_i : \lim_{\mathcal{I}} F \rightarrow F(i)) \in \mathcal{C}_1 \mid i \in \mathcal{I}_0\}$, such that $\forall (f : i \rightarrow j) \in \mathcal{I}_1 : \phi_j = F(f) \circ \phi_i$, with the **universal property** that, whenever there is an object $X \in \mathcal{C}_0$ with arrows $\{(g_i : \lim_{\mathcal{I}} F \rightarrow F(i)) \in \mathcal{C}_1 \mid i \in \mathcal{I}_0\}$ such that $\forall (f : i \rightarrow j) \in \mathcal{I}_1 : g_j = F(f) \circ g_i$, there is a unique arrow $u : \lim_{\mathcal{I}} F \rightarrow X$ such that $\forall i \in \mathcal{I}_0 : g_i = \phi_i \circ u$. Limits are a generalisation of **products** and **equalisers** and are **dual** to the concept of a **colimit**.

METRIC SPACES

*Limit (in a Metric Space)***

SAM'S MATHS DICTIONARY

TOPOLOGY

*Limit (in a Topological Space)***

SAM'S MATHS DICTIONARY

LINEAR ALGEBRA

*Linearly Independent***

SAM'S MATHS DICTIONARY

Given a **metric space** (X, d) and a **sequence** $(a_i)_{i=1}^n$ in X , a point $L \in X$ is a limit of the sequence if, for any given radius, there is a point in the sequence beyond which all terms lie within that distance of L :

$\forall \epsilon > 0 : \exists N > 0 : \forall n > N : d(L, a_n) < \epsilon$. If a limit exists then it is unique, and it is also the limit in the **topological sense** when X is endowed with the **metric topology**.

Given a **topological space** (X, \mathcal{T}) and a **sequence** $(a_i)_{i=1}^n$ in X , a point $L \in X$ is a limit of the sequence if, for any **open neighbourhood** of L , there is a point in the sequence beyond which all terms lie within that neighbourhood: $\forall U \in \mathcal{T} : L \in U \Rightarrow (\exists N > 0 : \forall n > N : a_n \in U)$. In a general topological space, the limit is not necessarily unique.

Given a **vector space** V over a **field** k , a **subset** $S \subseteq V$ is called 'linearly independent' if the only finite linear combination of elements of S to equal 0 is the trivial one, i.e. $\forall T \subseteq S : |T| < \infty \Rightarrow (\forall [f : T \rightarrow k, v \mapsto a_v] : \sum_{v \in T} a_v v = 0 \Rightarrow \forall v \in T : a_v = 0)$ - for any finite subset T of S and assignment of coefficients in k to each element of T , if the sum of these vectors with these coefficients is the zero vector then all coefficients must have been 0. In the definition of a **Schauder basis**, the finiteness condition on T is dropped.

ALGEBRA

Local Ring

SAM'S MATHS DICTIONARY

ALGEBRA

Localization (of a Ring)

SAM'S MATHS DICTIONARY

TOPOLOGY

Manifold

SAM'S MATHS DICTIONARY

A ring R is called 'local' if it has a unique maximal ideal (this may be maximal among left- right- or two-sided ideals, all are equivalent).

The localization of a commutative ring $(R, +, \cdot)$ by a multiplicatively closed subset S containing 1 (i.e. $\forall s, t \in S : s \cdot t \in S, 1 \in S$) is the ring $S^{-1}R$ of all formal fractions $\{\frac{r}{s} | r \in R, s \in S\}$, subject to the standard rules of addition and multiplication of fractions, i.e. $\frac{r_1}{s_1} + \frac{r_2}{s_2} := \frac{r_1 \cdot s_2 + r_2 \cdot s_1}{s_1 \cdot s_2}, \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} := \frac{r_1 \cdot r_2}{s_1 \cdot s_2}$ - in this way, the rational numbers \mathbb{Q} are simply the localization of the integers \mathbb{Z} by the set of non-zero integers. If \mathfrak{p} is a prime ideal of R , its complement is (by definition) a multiplicatively closed subset, so one often refers to ' R localised at \mathfrak{p} ', meaning the localization $(R \setminus \mathfrak{p})^{-1}R$.

A topological space (X, \mathcal{T}) is called a manifold of dimension n if there exist open sets $\{U_i | i \in I\} \subseteq \mathcal{T}$ and maps $\{\phi_i : U_i \rightarrow \mathbb{R}^n | i \in I\}$ such that the (U_i) form an open cover of X , each $\phi_i(U_i)$ is open in the metric topology of \mathbb{R}^n , and each ϕ_i is a homeomorphism onto its image. Often one also requires that X be Hausdorff and second countable or paracompact.

ALGEBRA

Maximal Ideal

SAM'S MATHS DICTIONARY

ORDER THEORY

*Maximal/Minimal Element***

SAM'S MATHS DICTIONARY

METRIC SPACES

*Metric Space***

SAM'S MATHS DICTIONARY

Given a ring R , an ideal \mathfrak{m} of R is called ‘maximal’ if it is a proper ideal ($\mathfrak{m} \neq R$) which is not contained in any other proper ideal. In the case of a non-commutative ring, one considers maximal left- and maximal right-ideals.

Given a partial order (P, \leq) , a maximal (resp. minimum) element of P is some $M \in P : \forall x \in P : x \leq M$ (resp. $m \in P : \forall x \in P : m \leq x$).

A metric space (X, d) consists of a set X (the ‘space’) with a function $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ (the ‘metric’) such that: d is symmetric, $\forall x, y \in X : d(x, y) = d(y, x)$, non-degenerate $\forall x, y \in X : d(x, y) = 0 \Leftrightarrow x = y$, and obeys the triangle inequality: $\forall x, y, z \in X : d(x, y) + d(y, z) \geq d(x, z)$.

TOPOLOGY

*Metric Topology***

SAM'S MATHS DICTIONARY

TOPOLOGY

*Metrizable***

SAM'S MATHS DICTIONARY

GROUP THEORY

*Monoid***

SAM'S MATHS DICTIONARY

Given a **metric space** (X, d) , the metric **topology** \mathcal{T}_d on X has as its open sets precisely those sets U such that every point of U is contained in an **open ball** contained in U , i.e. $\mathcal{T}_d := \{U \subseteq X \mid \forall x \in U : \exists r_x > 0 : B_{r_x}(x) \subseteq U\}$.

A **topological space** (X, \mathcal{T}) is called metrizable if the topology arises from a metric, i.e. if there exists a function $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ such that (X, d) is a **metric space**, and \mathcal{T} is equal to the **induced topology** on X due to d .

A monoid (M, \cdot) consists of a **set** M with a **binary operation** $\cdot : M \times M \rightarrow M$ which is **associative** and **unital** but, unlike for a **group**, not necessarily **invertible**. Examples include the natural numbers under addition and the integers under multiplication.

CATEGORY THEORY

*Natural Transformation***

SAM'S MATHS DICTIONARY

TOPOLOGY

*Neighbourhood***

SAM'S MATHS DICTIONARY

METRIC SPACES

*Open Ball***

SAM'S MATHS DICTIONARY

Given parallel **functors** $F, G : \mathcal{C} \rightarrow \mathcal{D}$ between **categories** \mathcal{C}, \mathcal{D} , a ‘natural transformation’ $\eta : F \Rightarrow G$ is an assignment of an arrow $(\eta_x : F(x) \rightarrow G(x)) \in \mathcal{D}_1$ to each object $x \in \mathcal{C}_0$, in a ‘natural way’, i.e. for every arrow $(f : x \rightarrow y) \in \mathcal{C}_1$ there is a commutative square in \mathcal{D} : $\eta_y \circ F(f) = G(f) \circ \eta_x$. Natural transformations (combined with **vertical composition** turn the **hom-set** $\mathbf{Cat}(\mathcal{C}, \mathcal{D})$ of functors from \mathcal{C} to \mathcal{D} into a category of its own, while vertical and **horizontal** composition together turn \mathbf{Cat} into a **2-category**.

Given a **topological space** (X, \mathcal{T}) , a neighbourhood of a point $x \in X$ is a **subset** $A \subseteq X$ which contains an open set containing x ; *exists* $U \in \mathcal{T} : x \in U \subseteq A$. Often when one talks of a neighbourhood of x , they just mean an open set containing x .

Given a **metric space** (X, d) , a point $x \in X$ and a radius $r \in \mathbb{R}_{>0}$, the open ball about x of radius r is the **subset** of all points of X closer to x than r ;

$$B_r(x) := \{y \in X \mid d(x, y) < r\}.$$

TOPOLOGY

*Open Cover***

SAM'S MATHS DICTIONARY

TOPOLOGY

Paracompact

SAM'S MATHS DICTIONARY

ORDER THEORY

*Partial Order***

SAM'S MATHS DICTIONARY

Given a **topological space** (X, \mathcal{T}) and a **subset** $A \subseteq X$, an open cover of A is a collection of open sets whose **union** contains A , i.e.

$$\{U_i\}_{i \in I} \subseteq \mathcal{T} : \cup_{i \in I} U_i \supseteq A.$$

A **topological space** (X, \mathcal{T}) is paracompact if every **open cover** of X has a **refinement** which is locally finite, i.e. given $\{U_i | i \in I\} \subset \mathcal{T}$ such that $\cup_{i \in I} U_i = X$, there exists a collection of open sets $\{V_j | j \in J\} \subseteq \mathcal{T}$ such that

$$\cup_{j \in J} V_j = X, \forall j \in J : \exists i \in I : V_j \subseteq U_i, \text{ and}$$

$\forall x \in X : \exists \mathcal{N}_x \in \mathcal{T} : x \in \mathcal{N}_x \wedge \#\{j \in J | \mathcal{N}_x \cap V_j \neq \emptyset\}$ (a refinement of an open cover is a second cover such that each open set of the refinement is contained in at least one of the sets of the original cover and is locally finite if, for each point in the space, there is some **neighbourhood** of the point such that a finite number of sets in the refinement intersect the neighbourhood).

Paracompactness is a weaker property than **compactness**, and is often a required property of **manifolds** as every open cover of a paracompact space admits a **partition of unity**.

A partial order (P, \leq) is a **set** P equipped with a **relation** \leq which is **reflexive**, **antisymmetric**, and **transitive**, e.g. $(\mathcal{P}(X), \subseteq)$, where X is any **set** and $\mathcal{P}(X)$ is its **power set**.

SET THEORY

*Partition***

SAM'S MATHS DICTIONARY

SET THEORY

*(Axiom of the) Power Set***

SAM'S MATHS DICTIONARY

CATEGORY THEORY

*Product (Category Theory)***

SAM'S MATHS DICTIONARY

A partition of a set X is a collection $P \subseteq \mathcal{P}(X)$ of **non-empty subsets** of X such that their **union** is all of X , $\cup P = \cup_{A \in P} A = X$ and they are pairwise disjoint: $\forall A, B \in P : A \cap B = \emptyset \Leftrightarrow A \neq B$. Equivalently, each element of x is contained in a unique element of P , $\forall x \in X : \exists! A \in P : x \in A$.

Given a set X , its ‘power set’ $\mathcal{P}(X)$ is the set of all **subsets** of X ; $\mathcal{P}(X) = \{U \mid U \subseteq X\}$. The ‘axiom of the power set’, one of the **Zermelo-Fraenkel** axioms, asserts that for any set X , there exists a set whose elements are precisely the subsets of X (i.e. the power set), in symbols:

$$\forall X : \exists \mathcal{P}(X) : \forall U : (U \in \mathcal{P}(X) \iff \forall a : a \in U \Rightarrow a \in X).$$

Given a **category** \mathcal{C} and a collection $(A_i)_{i \in I}$ of objects in \mathcal{C} , a product of the A_i is an object $\prod_{i \in I} A_i$ of \mathcal{C} and for each $i \in I$ an arrow $\pi_i : \prod_{j \in I} A_j \rightarrow A_i$ satisfying the following **universal property**: for any object X of \mathcal{C} with a collection of arrows $(f_i : X \rightarrow A_i)_{i \in I}$, there exists a unique arrow $(\prod_{i \in I} f_i) : X \rightarrow \prod_{i \in I} A_i$ such that $\forall i \in I : \pi_i \circ (\prod_{i \in I} f_i) = f_i$. If a product exists, then it is unique up to unique **isomorphism** (if $(X, (\pi_i)_{i \in I})$ and $(Y, (\phi_i)_{i \in I})$ are both products of $(A_i)_{i \in I}$ in \mathcal{C} , then there exists a unique arrow $u : X \rightarrow Y$ such that $\forall i : \pi_i = \phi_i \circ u$, and in fact u must be an isomorphism. The categorical product is the prototypical example of a **limit**.

TOPOLOGY

*Product Topology***

SAM'S MATHS DICTIONARY

SET THEORY

*Proposition***

SAM'S MATHS DICTIONARY

SET THEORY

*Quotient Set***

SAM'S MATHS DICTIONARY

Given two **topological spaces** (X, \mathcal{T}_X) , (Y, \mathcal{T}_Y) , one defines the product topology on their **Cartesian product** $X \times Y$, $\mathcal{T}_{X \times Y}$, as the **coarsest** topology on $X \times Y$ to contain all sets $U \times V$ where $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$, i.e. the topological space with **basis** $\{U \times V \subseteq X \times Y \mid U \in \mathcal{T}_X \wedge V \in \mathcal{T}_Y\}$. The space $(X \times Y, \mathcal{T}_{X \times Y})$ is the product of the aforementioned spaces in the **categorical sense** (with projection maps $\pi_1 : X \times Y \rightarrow X, (x, y) \mapsto x$, $\pi_2 : X \times Y \rightarrow Y, (x, y) \mapsto y$).

In metamathematics, a proposition is a statement with definite truth value (in classical logic the truth values are simply ‘True’ and ‘False’, but in other models of nonstandard logic the truth values could be e.g. any element of $[0, 1]$, or of a more general **Boolean algebra**).

Given a **set** X and an **equivalence relation** \sim on X , the quotient of X by \sim is the set of all equivalence classes of X under \sim , i.e. $X / \sim := \{[x] \subseteq X \mid x \in X\}$.

SET THEORY

*Reflexive***

SAM'S MATHS DICTIONARY

SET THEORY

*Relation***

SAM'S MATHS DICTIONARY

SET THEORY

*(Principle of) Restricted Comprehension***

SAM'S MATHS DICTIONARY

A **relation** \mathcal{R} is reflexive on a **set** X if each element of X is related to itself,

$$\forall x \in X : x\mathcal{R}x.$$

A relation is a **proposition** of two variables, where typically each variable may belong to a fixed **set** or **class**; for R a relation on X, Y we typically write this proposition as either $R(x, y)$ or xRy . One often identifies a relation R with its **graph** $\Gamma_R = \{(x, y) \in X \times Y | xRy\}$, and if Y is not specified one interprets ‘ R is a relation on X ’ as $\Gamma_R \subseteq X \times X$.

The principle of restricted comprehension states that, for any **set** x and **predicate** P , there exists a set (written $\{a \in x | P(a)\}$) whose elements are precisely those elements a of x which satisfy $P(a)$:

$$\forall x : \exists y : \forall a : (a \in y \Leftrightarrow a \in x \wedge P(a)).$$
 This is an axiom schema, as a copy of it applies for every predicate P , and it is implied by the stronger **axiom of replacement**: take a set x and a predicate P , then either there is no element of x satisfying P (and so the required set is \emptyset), or there is some $\tilde{a} \in x$ such that $P(a)$. In the second case, one forms a relation \mathcal{R} by

$$a\mathcal{R}b \Leftrightarrow a \in x \wedge ((P(a) \wedge a = b) \vee (\neg P(a) \wedge b = \tilde{a}))$$
 which is **functional** on x , and the image of each $a \in x$ is either a itself (if $P(a)$ holds) or \tilde{a} (if $P(a)$ is false), so the image of x under \mathcal{R} is precisely those elements of x for which P is true. A stronger version of the principle of restricted comprehension which is not an axiom of set theory (as it leads to inconsistencies) is the principle of unrestricted comprehension, which would allow for sets of the form $\{x | P(x)\}$.

ALGEBRA

*Ring***

SAM'S MATHS DICTIONARY

GROUP THEORY

Semigroup

SAM'S MATHS DICTIONARY

LIE ALGEBRAS

*Semisimple Lie Algebra***

SAM'S MATHS DICTIONARY

A (unital) ring $(R, +, \times)$ consists of a **set** R with **binary operations** $+, \times : R \times R \rightarrow R$ ('addition' and 'multiplication') such that $(R, +)$ is an **abelian group** with identity 0_R , (R, \times) is a **monoid** with identity 1_R , and \times **distributes** over $+$, i.e

$\forall a, b, c \in R : (a + b) \times c = a \times c + b \times c, a \times (b + c) = a \times b + a \times c$. One generally also requires that $0_R \neq 1_R$. Sometimes a ring is not required to have a multiplicative identity, so that R, \times is instead a **semigroup**.

A semigroup (S, \cdot) consists of a **set** S with a **binary product** $\cdot : S \times S \rightarrow S$ which is **associative**.

A **Lie algebra** L is called 'semisimple' if it has **ideals** $(\mathfrak{g}_i)_{i \in I}$ such that each \mathfrak{g}_i is **simple** and L is their **direct sum**, $L = \bigoplus_{i \in I} \mathfrak{g}_i$.

TOPOLOGY

*Sequence***

SAM'S MATHS DICTIONARY

SET THEORY

*Set***

SAM'S MATHS DICTIONARY

LIE ALGEBRAS

*Simple Lie Algebra***

SAM'S MATHS DICTIONARY

Given a **set** X (which may additionally be a **topological space** or **metric space** or some other structure with an underlying set), a sequence in X is just a **function** $a : \mathbb{N} \rightarrow X, n \mapsto a_n$. It is common to denote a sequence by its terms, $(a_i)_{i=0}^{\infty}$, instead of referring to the function itself.

Loosely speaking, a set is a collection of elements, denoted $a \in X$ (' a is contained in the set X '). One may denote a set by a rule which its members obey: $X = \{a \in Y | P(a)\}$, for Y some larger set and $P(a)$ some proposition. More formally, a set is an object which obeys a set of axioms, typically the **Zermelo-Fraenkel axioms** and the **axiom of choice**, which tell us what is and is not a set.

A **Lie algebra** L is called 'simple' if it is not **abelian** and also has no **ideals** other than L and 0 .

LINEAR ALGEBRA

*Span***

SAM'S MATHS DICTIONARY

ALGEBRA

Spectrum (of a Commutative Ring)

SAM'S MATHS DICTIONARY

LIE ALGEBRAS

*Subalgebra***

SAM'S MATHS DICTIONARY

Given a **vector space** $V, +, \cdot$ over a field k , the span of a **subset** $M \subseteq V$, written $\text{span}_k(M)$ or $\langle M \rangle$, is the set of all finite linear combinations of elements of M , $\text{span}_k(M) = \{\sum_{r=1}^n a_r v_r \mid n \in \mathbb{Z}_{\geq 0}, a_r \in k, v_r \in M\}$. The span is also the smallest (by inclusion) **sub-vector space** of V to contain M or, equivalently, the **intersection** of all subspaces which contain M . If V is further a **topological vector space**, then the closure of the span, $\overline{\text{span}_k(M)}$, is the set of all **convergent** (possibly infinite) linear combinations of elements of M .

The spectrum of a **commutative ring** R , $\text{Spec}(R)$, is the **set** of all its **prime ideals**. The spectrum may be made into a **topological space** by endowing it with the **Zariski topology** - the closed sets are given by $V_I := \{\mathfrak{p} \in \text{Spec}(R) \mid I \subseteq \mathfrak{p}\}$, where I is any **ideal** of R . In this way, Spec is a **contravariant functor** from **CRing** to **Set**, as any **ring homomorphism** $\phi : R \rightarrow S$ induces a map $\mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$ which is **continuous** with respect to this topology.

Given an **algebra** (\mathfrak{A}, \diamond) over a ring R , a subalgebra of \mathfrak{A} is a **nonempty subset** $\mathfrak{B} \subset \mathfrak{A}$ such that (\mathfrak{B}, \diamond) is also an algebra, i.e. $\forall x, y \in \mathfrak{B} : \forall r \in R : x + y \in \mathfrak{B} \wedge rx \in \mathfrak{B} \wedge -x \in \mathfrak{B} \wedge x \diamond y \in \mathfrak{B}$.

GROUP THEORY

Subgroup

SAM'S MATHS DICTIONARY

SET THEORY

*Subset***

SAM'S MATHS DICTIONARY

ALGEBRA

*Subset Product (Algebra)***

SAM'S MATHS DICTIONARY

Given a **group** (G, \cdot) , a 'subgroup' of G is a **non-empty subset** $H \subseteq G$ such that (H, \cdot) is also a group, i.e. $\forall g, h \in H : g \cdot h \in H, e_G \in H, g^{-1} \in H$. One writes $H \leq G$, and calls H 'proper' if $H \neq G$ and 'non-trivial' if $H \neq \{e_G\}$.

A 'subset' of a **set** X is a set U whose elements are all elements of X . We denote this **relation** by $U \subseteq X$; $U \subseteq X \Leftrightarrow \forall a : a \in U \Rightarrow a \in X$. If $U \neq X$, then we call U a 'proper' subset of X .

Given an **algebra** (\mathcal{A}, \diamond) over a **field** k and two **subsets** $S, T \subseteq \mathcal{A}$, their product $S \diamond T$ is the **subspace** of \mathcal{A} **spanned** by all pairs $s \diamond t$ with $s \in S, t \in T$, i.e. $S \diamond T := \{\sum_{i=1}^n a_i (s_i \diamond t_i) | a_i \in k, s_i \in S, t_i \in T\}$.

TOPOLOGY

*Subspace Topology***

SAM'S MATHS DICTIONARY

SET THEORY

*Superset***

SAM'S MATHS DICTIONARY

ORDER THEORY

*Supremum***

SAM'S MATHS DICTIONARY

Given a **topological space** X with topology \mathcal{T} , any **subset** $A \subseteq X$ can be made into a topological space by the ‘subspace topology’ $\mathcal{T}|_A = \{A \cap U \subseteq A \mid U \in \mathcal{T}\}$, where the open subsets of A are the **intersections** of A with the open subsets of X .

A ‘superset’ of a **set** X is a set S which contains all elements of X . We denote this **relation** by $S \supseteq X$; $S \supseteq X \Leftrightarrow \forall a : a \in X \Rightarrow a \in S$. If $S \neq X$, then we call S a ‘proper’ superset of X .

Given a **Partial Order** (P, \leq) and a subset $A \subseteq P$, the supremum of A , if it exists, is the least upper bound of A , i.e. the element $\sup A \in P$ such that $\forall x \in P : (\forall a \in A : a \leq x \Rightarrow \sup A \leq x)$. If the supremum exists it must be unique.

SET THEORY

*Symmetric Relation***

SAM'S MATHS DICTIONARY

CATEGORY THEORY

*Terminal Object***

SAM'S MATHS DICTIONARY

MISCELLANEOUS

*Topological Field***

SAM'S MATHS DICTIONARY

A **relation** R on a **set** X is symmetric if $\forall x, y \in X : xRy \Rightarrow yRx$. A relation is antisymmetric if $\forall x, y \in X : (xRy \wedge yRx) \Rightarrow x = y$.

An object τ of a **category** \mathcal{C} is called 'terminal' if, for any other object x of \mathcal{C} , there is a unique arrow from x to τ , i.e.
 $\forall x \in \mathcal{C}_0 : \exists! \tau_x \in \mathcal{C}_1 : \text{dom}(\tau_x) = x \wedge \text{cod}(\tau_x) = \tau$ or (in terms of **hom-sets**)
 $\forall x \in \mathcal{C}_0 : \text{Hom}_{\mathcal{C}}(x, \tau) = \{\tau_x\}$. If a category has terminal objects they are all **isomorphic** to each other.

A topological field $(k, \mathcal{T}, +, \cdot)$ is a **topological space** (k, \mathcal{T}) such that $(k, +, \cdot)$ is a **field** and $\cdot, + : k \times k \rightarrow k$ are both **continuous** when $k \times k$ is endowed with the **product topology**.

TOPOLOGY

*Topological Space***

SAM'S MATHS DICTIONARY

FUNCTIONAL ANALYSIS

*Topological Vector Space***

SAM'S MATHS DICTIONARY

ORDER THEORY

*Total Order***

SAM'S MATHS DICTIONARY

A 'topological space' (X, \mathcal{T}_X) consists of a **set** X (the 'space') and a set $\mathcal{T}_X \subseteq \mathcal{P}(X)$ of **subsets** of X , the 'topology' on X , the elements of which are called the 'open subsets' of X . \mathcal{T}_X is required to satisfy the following: The total space and the **empty set** must be open: $X, \emptyset \in \mathcal{T}_X$; The **union** of any collection of open sets must be open: $\forall \mathcal{U} \subseteq \mathcal{T}_X : \cup \mathcal{U} \in \mathcal{T}_X$; The **intersection** of two (and hence any finite number of) open sets must be open:

$$\forall U_1, U_2 \in \mathcal{T}_X : U_1 \cap U_2 \in \mathcal{T}_X.$$

A topological vector space $(V, \mathcal{T}, +, \cdot)$ over a **topological field** k is a **k -vector space** $(V, +, \cdot)$ such that (V, \mathcal{T}) is a **topological space**, and the maps $+: V \times V \rightarrow V$ and $\cdot : k \times V \rightarrow V$ are both **continuous** when $V \times V$ and $k \times V$ are endowed with the **product topology**.

A **partial order** (P, \leq) is further a total order if any two elements are comparable, i.e. $\forall x, y \in P : x \leq y \vee y \leq x$.

SET THEORY

*Transitive***

SAM'S MATHS DICTIONARY

SET THEORY

*(Axiom of) Union***

SAM'S MATHS DICTIONARY

LIE ALGEBRAS

*Universal Enveloping Algebra***

SAM'S MATHS DICTIONARY

A **relation** R on a **set** X is transitive if $\forall x, y, z \in X : (xRy \wedge yRz) \Rightarrow xRz$.

Given a **set** X , the union of X , denoted $\cup X$, is the set which contains the elements of all the sets in X - in symbols: $a \in \cup X \Leftrightarrow \exists x \in X : a \in x$, or equivalently $\cup X = \{a | \exists x \in X : a \in x\}$. If $X = \{x_1, \dots, x_n\}$ consists of a finite number of sets then we often write the union of X as $\cup X = x_1 \cup \dots \cup x_n$. If $X = \{x_i | i \in I\}$ is an indexed collection of sets then we often denote the union as $\cup X = \cup_{i \in I} x_i$. The axiom of union, one of the axioms of **Zermelo-Fraenkel** set theory, states that for any set X , its union is also a set: $\forall X : \exists \cup X : \forall a : (a \in \cup X \Leftrightarrow \exists x : a \in x \wedge x \in X)$.

Given a **Lie algebra** \mathfrak{g} over a **ring** R , the universal enveloping algebra of \mathfrak{g} , $U(\mathfrak{g})$, is an **associative R -algebra** which contains \mathfrak{g} as a **sub algebra**. It is constructed by taking the **free associative R -algebra** over \mathfrak{g} , $R\langle \mathfrak{g} \rangle = A$, and taking the quotient by the **sub-module** spanned by:
 $\{x +_A y - (x +_{\mathfrak{g}} y), r \cdot_A x - (r \cdot_{\mathfrak{g}} x), xy - yx - [x, y] | r \in R, x, y \in \mathfrak{g}\}$.

LINEAR ALGEBRA

*Vector Space***

SAM'S MATHS DICTIONARY

CATEGORY THEORY

*Vertical Composition***

SAM'S MATHS DICTIONARY

SET THEORY

*Zermelo-Fraenkel Axioms of Set Theory***

SAM'S MATHS DICTIONARY

Given a **field** $(k, +, \times, 0_k, 1_k)$, a ‘vector space over k ’, or a ‘ k -vector space’ $(V, +, \cdot)$ is a **set** V , with a **binary operation** $+$: $V \times V \rightarrow V$ (‘vector addition’) such that $(V, +)$ is an **abelian group**, and an operation \cdot : $k \times V \rightarrow V$ (‘scalar multiplication’) which is **left-unital** (i.e. $\forall v \in V : 1_k \cdot v = v$), **associates** with multiplication in the field (i.e. $\forall a, b \in k : \forall v \in V : a \cdot (b \cdot v) = (a \times b) \cdot v$), and **distributes** over both scalar and vector addition

$$(\forall a, b \in k : \forall u, v \in V : (a + b) \cdot (u + v) = a \cdot u + a \cdot v + b \cdot u + b \cdot v).$$

Given three parallel **functors** $F, G, H : \mathcal{C} \rightarrow \mathcal{D}$ between **categories** \mathcal{C}, \mathcal{D} and two **natural transformations** $\eta : F \Rightarrow G, \xi : G \Rightarrow H$, their vertical composition is the natural transformation $(\xi \circ \eta) : F \Rightarrow H$ defined by

$$(\xi \circ \eta)_x = (\xi_x \circ \eta_x) : F(x) \rightarrow H(x).$$

The Zermelo-Fraenkel axioms of **set theory** (usually along with the **axiom of choice**) are a collection of 8 axioms on the nature of sets. The axioms are those of: **extensionality, the empty set, the pair set, unions, replacement, power set, infinity, and foundation.**

SET THEORY

*Zorn's Lemma***

SAM'S MATHS DICTIONARY

Given any **partial order** (P, \leq) such that every **chain** in P has an **upper bound**, P has a **maximal element**. This is equivalent to the **axiom of choice** and the statement "every **vector space** has a **basis**".